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REALATIVITY IN FINANCE: GOALS AND RISK-BASED ASSET PRICING FOR INVESTORS WITH MULTIPLE STOCHASTIC GOALS AND AGENTS

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This asset pricing model incorporates four positive realities of investing; that investors have many stochastic goals, seek to delegate to skillful agents, explicitly specify risk budgets, and maximize risk-adjusted relative returns. As a result, it also incorporates the relative nature of investing—"Realativity". Critical to investment practice, it provides asset pricing, asset allocations, and risk-adjusted performance measures that are consistent. Assets are priced with just two goal-replicating assets and the absolute risk-free asset. The pair-wise equilibrium model uses observable assets and risk budgets, offers practical asset allocation recommendations, and captures dual attributes of risky assets (i.e., risky asset and hedge for other goals). Furthermore, asset allocation is "view neutral" and does not require expected return forecasts which are notoriously incorrect. It also possibly explains other interesting investment phenomena—e.g., why two pension funds with similar risk budgets could have very different asset allocations or why their expected returns forecasts may differ.

"When it is obvious that the goals cannot be reached, don't adjust the goals, adjust the action steps." Confucius[1]

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1 Introduction: Three Facets of Investing and a New Positive Approach

Investors require a consistent asset pricing model, asset allocation recommendations, and risk-adjusted performance measures (or the "three facets of investing") to be effective in managing portfolios. This paper uses a new positive approach to derive a normative asset pricing and asset allocation model based on investors maximizing goal/benchmark-relative risk-adjusted returns for many goals. It begins with the four realities of investing and attempts to develop a robust theoretical framework to allow for further examination and improvement of investment techniques adopted by retail and institutional investors. Hence, we coin a new phrase — "Realativity" — as we incorporate reality and multiple levels of relativity into decision-making in finance. Moreover, recent innovations in financial markets, especially in Brazil, where instruments have been issued specifically for retirement (January 2023) and education (August 2023) based on related research of the author (Muralidhar, 2016), demonstrate many valuable practical implications of this theory.

Traditional academic theory, which is based on unobserved investing practices (e.g., investors maximizing utility or that investors only care about absolute wealth, while ignoring goals), results in recommendations that are difficult to implement or reconcile with what is observed in practice, often requiring nuances that are not easy for practitioners to comprehend. For example, Two Fund Separation (Tobin, 1958), where investors should split their assets between a "market" portfolio and a (absolute) risk-free asset based on their risk aversion, is a very appealing theoretical concept, but not observed in practice, especially since there is no clarity as to: (i) the exact constituents of an investible "market" portfolio, (ii) the utility function of a pension fund or endowment, or (iii) the exact specification of the risk aversion parameter by investors. Investors have tried to adapt this theory to liability-driven investing (LDI) where the portfolio is recommended to be split between the Liability Hedge (Tobin's "safe" asset) and the Growth Portfolio (Tobin's "market" portfolio) as in Muralidhar (2011).

The approach in this paper, instead, incorporates four key realities of investing. First, that individuals (and even institutions) invest to achieve many stochastic goals. Savings for each goal are often custodied in different accounts/custodians—ranging from retirement (Merton, 2007) in a 401(K), to saving for a child's college education (Muralidhar, 2016) in a 529 Plan, to saving for future health expenses in a Health Savings Account (HSA), among many others. By stochastic goal, we mean that the present value of the stream of cash flows required for the goal can change daily because of changes in market parameters, including interest rates, and various types of inflation (standard-of-living, tuition or health, respectively), or the occurrence of an event. Similarly, some institutional investors manage multiple portfolios under the same governance structure and team: endowments, one or more pension funds depending on the beneficiaries (academic versus support staff), retiree health benefits, and even insurance portfolios. Second, investors delegate investment decisions to agents (Brennan, 1993; Allen, 2001), but seek skillful agents (Ambarish and Seigel, 1996). Third, investors seek to maximize goal-relative risk-adjusted returns (Markowitz, 1952, 1990; Lintner, 1965; Modigliani and Modigliani, 1997; Muralidhar, 2000; Perold, 2004). And fourth, that they explicitly specify their risk budget in their investment policy statements (IPS) in terms of absolute and relative risk targets.[2]

In a stochastic, single-goal model, it has been shown that asset allocation is goal-centric (Sharpe and Tint, 1990; Merton, 2007), making asset pricing relative to the goal as well (Muralidhar *et al.*, 2014). A stochastic, multiple goal, asset pricing model, with associated asset allocation recommendations, especially with agency layered on top of it, raises unique challenges for a utility-based approach because of the problem of aggregating preferences.[3] But, approaching the problem from the perspective of maximizing risk-adjusted returns, which is typically how investors behave, offers an opportunity for a new solution and interesting implications for asset pricing, and asset allocation, that can potentially explain and improve current investment practice and theory.

The model is predicated on the assumption that for each stochastic goal considered, a goal-replicating asset, alternatively, the "relative safe asset" or benchmark replicating asset, exists or can be created. Muralidhar (2015) first made the case for such instruments to be created and this idea has been extended to retirement and other goals (Merton and Muralidhar, 2016; Muralidhar, 2016, 2019b).[4] Thereafter, it is easy to see that a relative safe asset for one goal is risky for another and vice versa as the cash flows are very different. This is an acceptable assumption. Following the recommendation in Merton *et al*. (2020), Brazil issued a bond, RendA+ (or "Retirement Income Extra"), in January 2023 to serve as the safe asset for retirement, with a very unique set of cash flows, customized specifically for the goal of ensuring effective retirement investing, and this has been very successful raising R1.2 bn in less than 5 months via 60,000 investors.[5] In August 2023, given the success of RendA+ and following the recommendations of Muralidhar (2016), Brazil issued EducA+ (or "Education Income Extra") to help improve the savings and investments for higher education. The *relative* safe asset is different from the *absolute* risk-free asset (typically proxied by a Treasury bill and Tobin's "safe" asset), which is the anchor to traditional asset pricing models. One can think of these relative safe assets as "Goals-based Arrow–Debreu securities". In a stochastic goals-based world, the absolute risk-free asset is also risky relative to the goal as protection of principal or wealth, embedded in the T-bill, is risky from a cash flow perspective for a retirement investor who wants guaranteed, real, retirement income for 20 plus years. Notice that the education relative risk-free asset is also risky for a retirement investor and vice versa, and therein lies the solution to the model.

As a result, assets from the goals-based perspective can be placed in three classes: (i) a single, *absolute* risk-free asset; (ii) multiple *relative*, risk-free assets or goal/benchmark replicating assets, depending on the number of goals; and (iii) multiple risky assets (all other assets). This classification is important because the asset pricing equations for each class are also unique. The resulting model, termed the Goals and Risk-based Asset Pricing Model (GRAPM), derived from optimal portfolio selection decisions based on at least two goals and clearly articulated risk budgets, provides a pair-wise equilibrium where risky assets can be priced solely with two goal-replicating assets and the absolute risk-free asset. It also provides a unique return for the absolute risk-free asset (which in traditional models has no anchoring value and is arbitrary), and key conditions for the relative risk-free assets in equilibrium. GRAPM provides consistent recommendations for the three facets of investing: asset pricing, asset allocation and risk-adjusted performance calculation.

Moreover, this novel approach appears to eradicate the "free parameter" problem in traditional models (Cochrane, 2005). The pricing model captures the unique duality of each risky asset; namely, that assets could have valuable hedging properties for one goal, while also serving as a risky asset for a second goal. This provides academic support to Coqueret *et al*. (2017), who argue for examining the use of a carefully constructed portfolio of equities (other than market-cap weighted indices) in hedging bond-like liabilities, effectively capturing the dual value of equities.

Muralidhar (2019a) shows that the many asset pricing models (based on habit, peers/home bias, agency, background risk, goals/liabilities, stochastic opportunity set, exclusion of arbitrage, the existence of stochastic discount factors, or unique securities) do not provide consistent recommendations for asset allocation and risk-adjusted performance, which together with asset pricing form the three key facets of investing for any practitioner. Few models, CAPM being one of the few, meet this high standard of providing consistent recommendations for the three facets. In summary, the results and implications are markedly different from any previous asset pricing model given the difference in approach and practical assumptions about investor behavior. For example, asset allocation decisions are completely independent of expected return forecasts, which Muralidhar (2011) has dubbed "View Neutral" as they do not require expected return forecasts.

GRAPM could be used to explain why, in practice, different investors have different expected return forecasts. As we show in Sections 2 and 3, if their forecast is based solely on their goal and not on the global equilibrium, then there could be differences across investors. This is most evident in examining the forecasts of the five consultants that advise the same New York City Bureau of Asset Management that has oversight for five different portfolios (with five different Boards).[6] One could rationalize this outcome as each fund under the Bureau having either a different goal and/or a goal-centric (as opposed to global) perspective on assets. Furthermore, forecasts of expected returns have been poor both in terms of level and direction (Housel, 2015; The Economist, 2017), and hence having a range of forecasts may be potentially more valuable for practitioners.[7]

Further, GRAPM also offers an explanation for why/how two investors (e.g., New Mexico Public Employees Retirement Association, or NMPERA, and Los Angeles County Employees' Retirement Association or LACERA), with identical specifications of absolute risk and tracking error, can have vastly different asset allocations.[8] This follows because they could have very different goals/goal-replicating portfolios, but also very different definitions of which asset they consider to be risky. GRAPM thus has the potential to benefit investors seeking to invest for multiple goals, with limited risk budgets that they seek to optimally exploit, by providing tailored investment recommendations. Finally, GRAPM has the potential to help academics develop a robust and practical heterogenous investor model, where each representative investor for a goal is one type of investor as opposed to requiring complex utility functions or specific market assumptions.

The present paper is structured as follows: Sections 2, 3, and 4 provide the models and method to derive GRAPM and discuss the key results. Section 2 reviews the M-square (Modigliani–Modigliani, 1997) and the M-cube (Muralidhar, 2000) risk-adjusted performance measures. M-cube will be used to derive GRAPM and Section 2 shows how M-cube provides optimal asset allocation recommendations to the absolute risk-free asset, the relative risk-free (hereafter used interchangeably with goal-replicating asset), and a generic risky asset for an investor with a stochastic goal and delegated implementation (to skillful agents). Section 3 derives GRAPM by using the M-cube measure across two goals and one generic risky asset, based on market equilibrium, to extract asset pricing recommendations. It highlights the multiple equilibrium equations that are derived from this "pair-wise" or "Janus" equilibrium approach and examines the practical implications of the equilibrium conditions. Section 4 introduces a third asset to show how this approach expands easily to multiple assets and provides new insights. Section 5 discusses these results and examines shortcomings and extensions, and Section 6 concludes.

2 M-Cube Risk-Adjusted Performance and Optimal Demand for Assets

GRAPM is based on a critical observation that investors articulate risk budgets in a very explicit manner: a clear absolute and relative volatility target.[9] It then uses the Lintner (1965)/Perold (2004) approach to derive normative asset pricing and asset allocation recommendations on the assumption that investors maximize risk-adjusted returns, with one key difference; Perold (2004) assumes that investors maximize the Sharpe Ratio (Sharpe, 1994), whereas this paper, in order to incorporate multiple, stochastic goals and agency, and a desire for skillful agents, assumes that investors maximize the M-cube risk-adjusted performance measure (Muralidhar, 2000). The M-cube measure, which extends the M-square measure (Modigliani and Modigliani, 1997), begins by assuming that investors split their portfolio among the absolute risk-free asset, the relative risk-free asset (or goal-replicating asset), and a risky asset to achieve the highest risk-adjusted relative return. The constraints on the absolute and relative volatility of the representative investor's portfolio (as in the market practice example noted in Endnote 9) leads to optimal allocations to each of these assets. This is an optimal solution, but not an equilibrium solution.

2.1 The M-square measure of risk-adjusted performance

The foundational papers of CAPM implicitly assume that investors are principals (i.e., do not delegate[10]) and have deterministic goals (Muralidhar, 2019a). As a result, the risk measure is absolute volatility and the normative asset allocation solution is that investors split their portfolio between the absolute risk-free asset (also the goal-replicating asset) and the market portfolio (i.e., the risky portfolio). The asset pricing model for risky assets, I, is similarly focused on the absolute risk-free asset (F) and the market portfolio (M), and depends on a single correlation and two volatilities (of the market portfolio and risky asset being priced), but with no constraints on these exogenous values—or the Cochrane (2005) "free parameter" problem. The CAPM-corresponding risk-adjusted performance measures, whether Sharpe (1994), GH1/GH2 (Graham and Harvey, 1994, 1997) or M-square (Modigliani and Modigliani, 1997), normalize the volatility of portfolios in different ways to ensure that risk-adjusted portfolios can be correctly compared. M-square is the most interesting for the pair-wise equilibrium approach examined below as it requires the investor to create a new portfolio, called the risk-adjusted portfolio (RAP), that levers the original "risky" asset or agent portfolio (using the absolute risk-free asset), to have the same volatility as that of the volatile benchmark. In other words, in an absolute/CAPM, only a single normalization—that of volatility—is needed as volatility is the measure of risk.

Modigliani and Modigliani (1997) assume that a principal has a benchmark that is replicated by asset L, and hires an agent, who creates an active portfolio, P. The principal cannot observe the true effort of the agent and can only monitor/forecast P's returns (and resulting volatility). If the agent is deceitful and the principal naïve (in that they only examine returns and not risk), then the agent has an incentive to lever P using the absolute risk-free asset (F), assuming that this can be done relatively easily/costlessly. Instead, Modigliani and Modigliani (1997) suggest that the principal should maximize the relative risk-adjusted excess return of the portfolio (through their own actions and the agent's portfolio), and not pay the agent for zero "intelligence" activities like leverage. This requires normalizing returns of P so that the volatility of the new risk-adjusted portfolio is identical to the volatility of the goal-replicating asset. This can be shown mathematically with a few simple equations.

Let A represent the risk-adjusted portfolio (or RAP) created using the active portfolio, P. Once risk-adjusted portfolio, A, and the benchmark, L, have the same volatility, the historical or expected returns of the A and L are comparable, providing excess risk-adjusted returns. In short, maximizing M-square risk-adjusted excess returns also provides consistent/optimal asset allocation recommendations in an absolute/CAPM world, but with recommendations that are independent of the assets' expect return, as opposed to Markowitz (1952). Mathematically, the M-square performance of the agent relative to the benchmark based on the goal can be established by the principal using the objective function specified in Equation (1), and based on two conditions shown in Equations (2) and (3). $r(*)$ measures the return of an asset or portfolio, σ is the volatility, and $E[*]$ is the expectations operator. F is the absolute risk-free asset with zero volatility and zero correlation to all other assets. For convenience, capital letters will be used to represent assets (with the exception of Tracking Error—TE), while Greek letters and lowercase letters will represent variables.

$$\text{Max } E[r(A) - r(L)] \tag{1}$$

$$\text{subject to } E[r(A)] = d \times E[r(P)] + (1 - d) \times r(F) \tag{2}$$

and

$$\sigma(A) = \sigma(L). \tag{3}$$

The solution to this maximization problem is $d = \sigma(L)/\sigma(P)$, and d measures the implicit or explicit leverage in portfolio P.[11] Since all agents have the same absolute risk (or volatility), Modigliani–Modigliani (1997) state that their M-square returns (or $r(A)$s) are now comparable. This is a non-equilibrium model (based on an implicit mean—variance utility function) so it is possible for $r(A)$s generated by portfolios of different agents to have different historical or expected returns, even though they all have the same volatility. This follows because the correlation of the A portfolios to the benchmark have not been normalized and hence differences in RAP can be attributed to differences in correlation between the agents' portfolios (P) and L (Muralidhar, 2000). Stated otherwise, while all portfolios have the same absolute risk, they can have very different risk relative to L (referred to as Tracking Error).

While consistent with CAPM, M-square, GH1/GH2 and Sharpe ratios do not appear to completely capture how investors actually budget risk. For example, two large and innovative pension funds (LACERA and NMPERA) have very explicit statements of objectives that are worth reviewing. While just two examples have been provided, these practices are widespread among institutional and some retail clients. NMPERA's IPS explicitly states that the Board established a 10.5% annualized target volatility for the strategic asset allocation (or $\sigma(L)$) and a 1.5% annualized tracking error (or TE hereafter) for all delegated decisions. We will refer to this specific tracking error budget as TE(Target) or τ.[12] The LACERA IPS states: "The Fund's long-term performance objective is to generate risk-adjusted returns that meet or exceed its defined actuarial target as well as its policy benchmark, net of fees, over the Fund's designated investment time horizon."[13] The actuarial target proxies the goal of this pension fund (i.e., the pension benefit payments). LACERA, like NMPERA, also articulates an explicit relative risk budget. Interestingly, despite having similar risk statements, the asset allocations (i.e., weights to different assets) of these two funds are very different and the assets in the two portfolios are also different and this will be explained later in Section 3.

2.2 *The M-cube measure of risk-adjusted performance*

Muralidhar (2000) extends the M-square risk-adjusted performance approach by arguing that in a "relative" paradigm, especially with a stochastic goal and agency, there are dual measures of risk: both absolute and relative risk (or TE); not just absolute risk as in M-square, GH1/GH2 or CAPM/Modern Portfolio Theory (MPT). Therefore, we require a dual normalization in a relative paradigm—of volatility and correlation. This is clearly shown in the case of LACERA and NMPERA, and many other investors. Therefore, an agent's portfolio (P) performance must be risk-adjusted, by levering/delevering using both F and the benchmark hedging asset, L, to create a new portfolio with both a target absolute volatility ($\sigma(L)$), as in M-square, and a target correlation to the benchmark (or *de facto* τ). The agency literature, in large part, has ignored this second critical constraint of normalizing correlations as well (Brennan, 1993; Cornell and Roll, 2005).[14] Hence, typical risk-adjusted measures like Sharpe, M-square or even GH1/GH2 cannot rank agents identical to rankings based on measures of confidence in skill (Ambarish and Seigel, 1996; Muralidhar, 2000). This is a very important point as principals would ideally want only skillful agents. M-cube ensures that all agents' risk-adjusted performance ($r(A)$s) have the same relative risk (TE) to L, but this must be achieved by ensuring that the risk-adjusted performance has been normalized to have the same volatility, $\sigma(L)$, and target correlation, $\rho(\tau, L)$. This dual normalization is important because a specific TE value can be achieved by different combinations of portfolio volatility and correlation relative to the benchmark.[15] But each combination of volatility and correlation, assuming a fixed (historical or expected) outperformance by an agent, implies very different levels of confidence that the performance is skill-based. It is easily shown that if two agents have the same expected return and tracking error, but with different volatilities and correlations, then the agent with the low volatility of P would be preferred to one with a high volatility of P (more noise). Appendix I derives the Ambarish–Seigel (1996) measure of confidence in skill to validate this point.

The M-cube approach, with a single stochastic goal, is also a non-equilibrium model like M-square. M-cube risk-adjusted returns are estimated by assuming that principals maximize expected funded status relative to the goal (i.e., value of assets divided by the value of the goal) as recommended in Merton (2007) and Sharpe and Tint (1990). Converting this goal into return terms, the investor seeks to maximize expected, goal-relative, risk-adjusted returns, $E[r(A) - r(L)]$, subject to three constraints highlighted in Equations (4)–(6). Interestingly, this treatment of the objective function is identical to Sharpe and Tint (1990), which shows that maximizing funded status reduces to maximizing $E[r(A) - r(L)]$ or Equation (1). However, now the risk-adjusted portfolio, A, is composed of the agent's portfolio, P, the goal-hedging portfolio (or relative risk-free asset), L, and F (absolute risk-free asset) as in Equation (4).

$$\begin{aligned} E[r(A)_{P,L|\tau}] = a^{P}_{\tau/L} \times E[r(P)] \\ + l^{L}_{\tau/L} \times E[r(L)] \\ + (1 - a^{P}_{\tau/L} - l^{L}_{\tau/L}) \\ \times r(F) \end{aligned} \quad (4)$$

$$\sigma(A) = \sigma(L) \quad (5)$$

and

$$\mathrm{TE}(A, L) = \mathrm{TE}(\text{Target}, L) \quad (6)$$

where $a^P_{\tau/L}$ is the allocation to the risky (agent) portfolio, P, given the goal of L and a target relative risk of T; $l^L_{\tau/L}$ is the allocation to the goal-hedging/goal-replicating asset, L (and measures what is *de facto* invested in the low-cost passive benchmark or "beta" in industry parlance). The superscript denotes the asset being allocated to; the subscript indicates the target relative risk (τ) and the goal-replicating asset (L). Similarly, $E[r(A)_{x,y|\tau}]$ is the expected return of A, assuming x is the risky asset, y is the goal, and τ is the target risk. We include these superscripts and subscripts in the formula because Sections 3 and 4 examine asset allocations based on multiple goals, risky assets, and goal-hedging assets to derive GRAPM. The balance of the assets is invested in the traditional absolute risk-free asset, F (and measures leverage). The term "*1-a-l*" is the corollary to "*1-d*" in M-square. Now

$$\mathrm{TE}(A, L) = \sqrt{[\sigma^2(A) - 2 * \rho(A, L) * \sigma(A) * \sigma(L) + \sigma^2(L)]} \tag{7}$$

where $\rho(*)$ is the correlation parameter. From the constraint on tracking error (Equation (6)), a unique target correlation between portfolio A and liability L, $\rho(\tau,L)$, is identified. Alternatively, instead of specifying a TE(Target, L), a principal could also specify $\rho(\tau, L)$. The derivation of M-cube is provided in Appendix II and the results are as follows:

$$a^P_{\tau/L} = \frac{\sigma(L)}{\sigma(P)}\left[\sqrt{\frac{[1-\rho(\tau, L)^2]}{[1-\rho(P, L)^2]}}\right] = \frac{\sigma(L)}{\sigma(P)}\left[\frac{\varphi(\tau, L)}{\varphi(P, L)}\right],$$

where

$$\varphi(I, J) = \sqrt{[1-\rho(I, J)^2]} \tag{8}$$

$$l^L_{\tau/L} = \rho(\tau, L) - \rho(P, L) \times \left[\sqrt{\frac{[1-\rho(\tau, L)^2]}{[1-\rho(P, L)^2]}}\right] = \rho(\tau, L) - \rho(P, L) \times \left[\frac{\varphi(\tau, L)}{\varphi(P, L)}\right]. \tag{9}$$

The most important observation from Equations (8) and (9) is that there is no expected return term in either a and l and hence the "view neutral" claim. To summarize, if $\rho(P, L)$ is the correlation of returns of L and P of each agent (which is easily calculated from actual data), and $\rho(\tau, L)$ as the target correlation,[16] which is specified by the principal, we can solve for the optimal asset allocation "a" as in Equation (8), and "l" in Equation (9). Unlike the traditional asset pricing approach, M-cube's asset allocation is influenced by a risk budgeting parameter that is easily stated/deduced; namely, $\rho(\tau, L)$. Also, in the CAPM case, there is no agent (or the principal does not care about the skill or tracking error of agents) and $\rho(\tau, L) = \rho(P, L)$. This gives us the M-square result; namely, $a = d$, and $l = 0$. So, M-square and CAPM are a very special case of M-cube and GRAPM with just a single normalization; and as shown in previous research, CAPM is a very special case of a Relative Asset Pricing Model (Muralidhar *et al.*, 2014).

From Equation (8), the allocation to risky assets is an increasing, linear function of the volatility of the goal, a decreasing nonlinear function of the target correlation (i.e., risk aversion), and increasing, nonlinear function to the correlation of the asset to the goal. Conversely, it is a decreasing function of its own volatility. Note that in the M-cube approach, the investor decides which asset(s) form the risky portfolio. As shown in Section 3, this freedom to choose what is "risky" is used to derive GRAPM and reflects reality. For example, one fund may decide that private credit

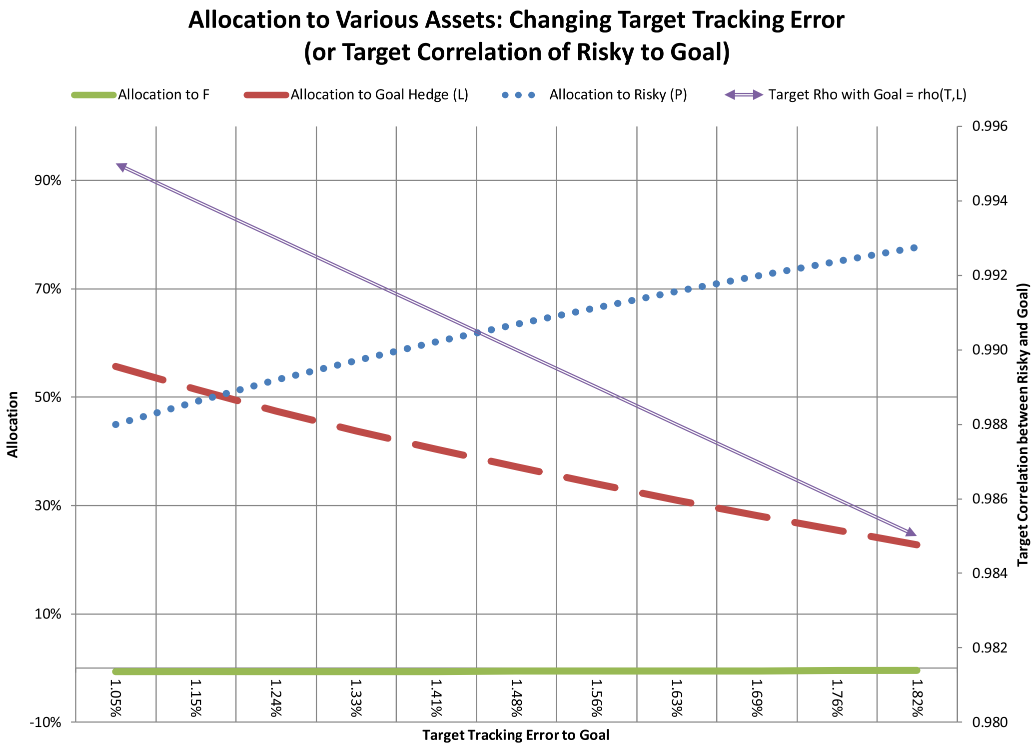

Figure 1 Allocation to risky assets, goal-replicating asset, and absolute risk-free asset for different target tracking errors.

is the risky asset they want to allocate to, and the second fund may decide that private equity is the risky asset of choice. As a result, these investors clearly violate Two Fund Separation and this approach may be a possible explanation for observed differences in portfolios and asset allocations across the investment universe and this is true of the asset allocations of NMPERA and LACERA referenced above, who have the same specification of risk budgets (and ostensibly similar goals).

Similarly, from Equation (9), the allocation to the goal-hedging asset is related positively and non-linearly to the target correlation, and negatively and non-linearly to the correlation of the risky asset to the goal. The allocation to the goal-replicating asset is independent of any volatility terms. These results are intuitively obvious as the allocation to risky assets should increase with increases in absolute and relative risk targets. Similarly, the allocation to the goal-hedging asset will increase, the lower the relative risk target. Figure 1 provides a visual of how one might establish a demand curve for these assets, based on this explicit measure of risk budget.

Assume $\sigma(L) = 10.5\%$, $\sigma(P) = 10.5\%$ and $\rho(P, L) = 0.99$. Then a $\text{TE(Target)} = 1.5\%$, implies that $\rho(\tau, L) \approx 0.995$. Figure 1 provides a range of tracking error targets around this TE(Target) by just varying $\rho(\tau, L)$. For low levels of relative risk (i.e., funded status or relative risk aversion is high), allocation to the liability hedge is high (dashed line) and declines thereafter as the allocation to risky assets (dotted line) rises. In this setting, the non-linearity is not as obvious as in the example in Muralidhar and Shin (2013) that examines likely allocations for a corporate pension fund, and hence a different goal-replicating asset (as corporate pension funds usually discount liabilities with a bond ladder with a duration of approximately 10–15 years depending on the maturity of the plan). The double-lined arrow provides the target correlation, $\rho(\tau, L)$, for these TE(Target) levels. The allocation to F ranges from -1% to 0% for these settings, but again is substantially different from zero in Muralidhar and Shin (2013). Interestingly, in January 2020, this technique was applied to the City of Austin's Employees' Retirement System (COAERS) and the results presented in Appendix III were remarkably close to the actual allocation for a target relative risk of 1.5% even though this approach had not been used by the Chief Investment officer.

Interestingly, "a" and "l" will feature in the asset pricing equation as a way of capturing the dual value of an asset in serving as a risky asset for some goals and hedging other goals, respectively. With these optimal asset allocation implications, based on the dual normalizations of volatilities and correlations, we can now articulate the pair-wise equilibrium to derive expected returns for all assets, but where there are limits to the expected return, correlation and volatility values that various assets (and classes of assets) can take (to address the "free parameter" issue of current models).

3 The "Pair-Wise" Equilibrium Approach (Two Relative Safe Assets)

To proceed from the demand curves to the asset pricing model we have to establish two equilibrium conditions. For the first condition, consider two optimal risk-adjusted portfolios for a given goal (first, using a generic risky asset, I; and second, using the second goal-replicating asset). The expected returns of these two risk-adjusted portfolios must be equal as they have identical risk characteristics to the goal.[17] This leads to one key equation to price I. But the same experiment can be repeated for the second goal as well, giving a second pricing equation for I. Free trading across investors with different goals (potentially "heterogeneous investors") forces a "pair-wise" or "Janus" equilibrium, as asset I can have only one expected return, ignoring market perfections, despite having different attractive attributes for each goal (i.e., as a goal hedge for one and/or as a risky asset for the other). As shown below, this reduces the model to a simple, global equilibrium equation such that all risky assets can be priced based on either goal-replicating asset. Furthermore, there are specific returns/relationships for F and the two goals. The pair-wise equilibrium, unlike say the CAPM, also requires correlations among (risky and relative safe) assets and volatilities of assets to adhere to very specific conditions.

Equation (4) shows that the goal-replicating asset L, F, and any risky asset can be used to create risk-adjusted portfolios with specific target volatility and correlation characteristics relative to that goal. With some reasonable assumptions and basic steps, we can derive the key equations for the pair-wise equilibrium.

3.1 Assumptions

The following assumptions are required for the model:

(i) Assume a one-period setting.
(ii) Investors/principals have a stochastic goal and delegate to one (or more) agent(s) to achieve that goal.
(iii) Principals seek the highest goal-relative, expected risk-adjusted excess return. If two portfolios have the same absolute and relative volatility, they would prefer the

portfolio with the higher expected risk-adjusted return.

(iv) Principals hire agents to take risk relative to their benchmark for one of the three reasons: the principal is underfunded[18]; the principal is incapable of managing assets[19]; or agents claim to have skill in outperforming the benchmark. Principals permit limited relative risk budgets as they are uncertain about the skill of agents and can only observe past returns, not effort or skill. They could extrapolate from past returns to forecast expected returns.

(v) Principals specify risk targets as shown in Section 2—with a specific absolute risk level, equal to the risk (or volatility) of the relative safe asset as in Equation (5),[20] and a relative risk budget for their agents, τ, as in Equation (6).

(vi) There are two goals, $G1$ and $G2$, respectively, each with representative investors for that goal. Initially, assume that the representative investors across goals $G1$ and $G2$ are unable to trade with one another to establish the optimal equations for asset pricing. We term this goal-specific or "myopic" expected returns. Then, they are permitted to trade assets with each other to ensure a global equilibrium across all goals, eliminating any inconsistencies in pricing assets using the myopic approach.

(vii) Both these goals have unique goal-replicating assets, $G1$ and $G2$, respectively.[21]

(viii) Assume F exists. Also, there exists the generic risky asset, I, that we seek to price. I is different from $G1$, $G2$, and F, and the absolute value of the correlation of I to $G1$ and $G2$ is different from unity.[22]

(ix) There is sufficient supply of all assets (and ignore supply-side issues) as this model is based on purely demand/asset allocation decisions.

(x) Assume agents can invest in just a single risky asset to create the delegated portfolio relative to the goal. They invest either in I or the second goal-replicating asset. We relax this assumption in Section 5.1.

(xi) Principals use M-cube risk-adjusted expected returns to evaluate agents.

(xii) For both goals, $G1$ and $G2$, the target correlation or $\rho(\tau, L)$ is identical. The τ need not be identical as that also depends on the volatility of the goal. This just simplifies the formulas, not the end result, and is easily relaxed later.

(xiii) There are no transactions costs and unrestricted trading prevents arbitrage.

(xiv) There are equal proportions of investors for both goals (or with identical assets under management) so that the initial/simple model is free of the fraction of investors for each goal. This is relaxed later in Section 5.1.

3.2 The Janus equilibrium solution

Eight major steps are needed to establish the "pairwise" asset pricing equations and are covered in Appendix IV. We can drop the τ subscript as the equations are independent of τ. Recall that there are three classes of assets, and GRAPM provides specific expected returns for each asset. The resulting returns/expected returns for the three classes of assets are as follows:

If we define

$$\mathrm{X}_{I,G1,G2} = \varphi(I, G1) - \rho(I, G2) \times \varphi(I, G1) + \rho(G1, G2) \times \varphi(I, G2) \qquad (10)$$

$$Y_{I,G1,G2} = \varphi(I, G2) - \rho(I, G1) \times \varphi(I, G2) + \rho(G1, G2) \times \varphi(I, G1). \qquad (11)$$

Then for the absolute risk-free asset,

$$r(F)_{G1,G2} = \frac{\left\{\begin{array}{l} E[r(G1) - \frac{\sigma(G1)}{\sigma(G2)} \\ \quad \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times E[r(G2)] \end{array}\right\}}{\left[1 - \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right)\right]}. \quad (12)$$

For the goal-replicating assets, Equation (13) must hold,

$$E[r(G1) - r(F)] = \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times E[r(G2) - r(F)]. \quad (13)$$

And finally, for all risk assets, the following equation would apply:

$$E[r(I) - r(F)] = \mathrm{Z}_{I,G1/G2} \times E[r(G1) - r(F)] \quad (14)$$

where Zeta is a form of a "covariance term" as beta is in CAPM, defined as

$$\mathrm{Z}_{I,G1/G2} = \frac{\sigma(I)}{\sigma(G1)} \times \left\{\left[\frac{\varphi(I, G2)}{\varphi(G1, G2)}\right] + \left(\frac{\mathrm{Y}_{I,G1,G2}}{X_{I,G1,G2}}\right) \times \rho(I, G2) - \left(\frac{Y_{I,G1,G2}}{X_{I,G1,G2}}\right) \times \rho(G1, G2) \times \left[\frac{\varphi(I, G2)}{\varphi(G1, G2)}\right]\right\}. \quad (15)$$

Notice, that Zeta is a function of two volatilities, $\sigma(I)$ and $\sigma(G1)$ (as in CAPM), but three correlations: $\rho(G1, G2)$, $\rho(I, G2)$, and $\rho(I, G1)$ One other key point to note, Equation (14) cannot be used for assets that are goal-replicating; instead, Equation (13) must be used for goal-replicating assets. Furthermore, since X and Y are functions of $\varphi(IJ)$, Zeta is not defined when $\rho(G1, G2)$, $\rho(I, G2)$ or $\rho(I, G1) = 1$, unless the other correlations are = 0.[23]

Interestingly, Zeta can be restated in terms of "a" and "l" using Equations (8) and (9) as follows and will be interpreted later:

$$\mathrm{Z}_{I,G1/G2} = \frac{a^{G1}_{\tau/G2}}{a^{I}_{\tau/G2}} + \frac{\sigma(I)}{\sigma(G1)} \times \left(\frac{Y_{I,G1,G2}}{X_{I,G1,G2}}\right) \times l^{G2}_{I/G2}. \quad (16)$$

Similarly, express $E[r(I)]$ as a function of $E[r(G2)]$ as the other risky asset, $r(F)$, volatilities and correlations (assuming $G1$ is the goal) as in Equation (17).

$$E[r(I) - r(F)] = \mathrm{Z}_{I,G2/G1} \times E[r(G2) - r(F)]. \quad (17)$$

We can see these dual equations in (14) and (17) for I as a pair-wise or "Janus" equilibrium.[24] Interestingly, given Equations (17), (13), and (14), there is a unique relationship between $\mathrm{Z}_{I,G1/G2}$ and $\mathrm{Z}_{I,G2/G1}$; namely, that

$$\mathrm{Z}_{I,G2/G1} = \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times \mathrm{Z}_{I,G1/G2}. \quad (18)$$

3.3 *Implications of GRAPM*

There are many interesting implications from this approach and model.

(i) GRAPM delivers the three facets—an asset pricing, asset allocation, and M-cube as the risk-adjusted performance measure—in one consistent model, while incorporating the four realities of relative investing.

(ii) With two goal-replicating assets and the absolute risk-free rate and a single risky asset, the model requires five critical equations: Equations (14) and (17) to price asset I, and Equations (12), (13), and

(16) to ensure that many "free parameters" are eliminated. All assets used to price asset I have very specific returns, correlations, and volatilities that must satisfy these additional equations.[25]

(iii) In this two-goal case, a generic asset I can be priced with both goal-replicating assets: using either $G1$ or $G2$. These assets, $G1$ and $G2$, can be easily observed compared to challenges faced in testing single or multiple factor asset pricing models. With Brazil issuing both a retirement bond in January 2023 and an education bond in August 2023, it will be possible to suggest prices of other assets with this model. Also, the equations to price goal-replicating assets are different from the equations to price generic risky assets, and the risk-free asset.

(iv) The pair-wise equilibria, resulting in multiple equilibrium equations, is not a shortcoming of the model, but rather a feature. With three relative safe assets, I can be priced using six combinations of $G1$, $G2$, and $G3$, as shown in Section 4, with additional interesting equilibrium conditions.

(v) Alternatively, if other conditions/market imperfections prevent a "general" equilibrium, we can derive a range of expected returns for each asset—a more reasonable assumption than having a single global point estimate (assuming everyone has the same expectation). Interestingly, when one compares the forecasts of consultants who advise institutional or retail clients, there can be wide dispersion in their forecasts and one explanation is that they are only looking at the forecast from the bias of their goals and not from a global equilibrium that these equations capture.

(vi) The Zeta term is in the spirit of the beta term of MPT, and relative beta of RAPM. Zeta, in Equation (16), captures the value of an asset in not only hedging the goals (i.e., the "l" variable), but also in serving as a risky asset as shown in the "a" variables—that potentially earns a higher return than the goal. This explains why Zeta is a unique function of asset allocation to risky assets and the goal-hedging asset. The first term is the ratio of allocations to the two risky assets for each goal, given the target risk, and the second term has the hedging component.

(vii) In short, the equilibrium expected returns of an asset is explicitly impacted by all assets—whether they are substitutes or complements—much like one experiences in financial markets.[26]

(viii) The final asset pricing model is independent of $\rho(\tau, L)$, but, in the two-goal case, depends on three correlations to be precise: $\rho(I, G1)$, $\rho(I, G2)$, and $\rho(G1, G2)$. With more goal-replicating assets, the basic asset pricing formula is unchanged as shown in Section 4.

(ix) Allocation recommendations are intuitive as discussed in Section 3.

$X(I, G1, G2)$	0.4741
$Y(I, G1, G2)$	0.6169
$X(I, G1, G2)/Y(I, G1, G2)$	0.7685
$Y(I, G1, G2)/X(I, G1, G2)$	1.3013

In summary, using the traditional absolute risk-free asset, F, and two goal-specific risk-free assets (say $G1$ and $G2$) allows one to triangulate to establish the return of any other risky asset, I. $G1$ and $G2$ individuals who are maximizing risk-adjusted returns subject to clearly articulated risk budgets will each have a unique demand for the risky asset, but the global equilibrium conditions across all goals ensures that returns (and other parameters) will stabilize at a level compatible with all goals. In a similar fashion, having heterogeneous investors (e.g., one focused on retirement, and one focused on saving for a child's college education), who are supplied their goal-specific risk-free asset and seek to maximize the relative risk-adjusted performance of their portfolios, is sufficient to solve out the asset pricing model. In a heterogeneous investors model, $G1$ individuals are one type of investor, and $G2$ individuals are a different type of investor, and it is clear that risky assets can be priced based on either goal, with useful asset allocation recommendations for all investors.[27]

4 The Three Goal-Replicating Asset Case (and a Test Simulation)

Adding a third goal/goal-replicating asset demonstrates the value of the GRAPM model as it further highlights how GRAPM eliminates "free parameters" that plague the CAPM Model (Cochrane, 2005) and allows for a generalization of the model to N assets.

4.1 Key equations

If we add a third goal and goal-replicating asset, $G3$, then four additional conditions (see Appendix V) must also hold for asset I, over and above equations (14) and (17). In short, there are now six potential equations for the expected returns of asset I, and, in equilibrium, all must provide the same result because all assets are linked by the "pair-wise" equilibrium. The key results are that:

$$Z_{I,G2/G1} = Z_{I,G2/G3} \tag{19}$$

$$Z_{I,G3/G1} = Z_{I,G3/G2} \tag{20}$$

$$Z_{I,G1/G2} = Z_{I,G1/G3} \tag{21}$$

$$\left(\frac{X_{I,G1,G3}}{Y_{I,G1,G3}}\right) = \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times \left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right). \tag{22}$$

And

$$\left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right) = \left(\frac{Y_{I,G3,G2}}{X_{I,G3,G2}}\right). \tag{23}$$

And similarly for the other combination of $G1$, $G2$, and $G3$.

4.2 A simple simulation with three assets

Assume the following assumptions on volatility and correlation for and among F, $G1$, $G2$, $G3$ as in Table 1 and also risky asset I. For the given, parameters, we tried to solve just for the expected return values of F and I that would solve the model while holding all other values constant to simplify the problem. The expected returns is as per column 2, which also nearly satisfies all the other conditions as this solution is off by 6 bps because the correlations, etc., were bluntly imposed on the model as opposed to being

Table 1 Estimating expected returns given estimates of volatilities and correlations—The three relative safe asset case.

ASSET PRICING								
	Expected Return	Volatility	Return/Risk	Correlation				
Asset				F	I	G1	G2	G3
F	1.79%	0		1	0	0	0	0
I	8.75%	10.0%	0.87	0	1	0.5	0.7	0.6
G1	5%	6.4%	0.78	0		1	0.3	0.4
G2	7%	8.0%	0.86	0			1	0.2
G3	5%	7.0%	0.71	0				1

Table 2 Testing the equilibrium value for I using the Zeta values.

Z(I, G2 = risky; G1 = goal)		1.3552
r(I) using G1 as goal	8.72%	
Z(I, G1 = risky; G2 = goal)		2.1364
r(I) using G2 as goal	8.65%	
Equilibrium	0.06543456924%	NO

Table 3 Testing the equilibrium value for I using the expanded form.

$r(I) - r(F)$	6.96%				
$r(G1) - r(F)$	3.21%				
$r(G2) - r(F)$	5.11%				
$r(G3) - r(F)$	3.21%				
Test of Expected Returns		r(G1) – r(F) LHS	r(G1) – r(F) RHS	Equilibrium	NO
$r(G1) - r(F) = [\{\sigma(G1)/\sigma(G2)\} * \{X/Y\} * \{r(G2) - r(F)\}]$		3.21%	3.15%	0.07%	

jointly optimized and will be examined in future research. Other values needed for the solution include

In testing whether all values were reconciled, we see a small discrepancy in Tables 2 and 3, but the results suggest that this could easily be reconciled if the optimization had been more general.

4.3 Implications of additional goal-replicating assets

Adding more goal-replicating assets gives not only more equilibrium equations for the generic risky asset I, but also additional equalities for Zeta and relationships between various Xs and Ys. What this implies is that the equilibrium has to be very compact across all variables; namely, that each asset in the market probably can only take limited expected returns, volatilities, and correlations to ensure this full market equilibrium. Hence our claim that there are probably few "free" parameters in this model. As shown mathematically in Appendix VI, we can also conclude that,

(i) For each risky asset I, if there are N goal-replicating assets, then there will be $N \times (N-1)$ equations for I.
(ii) There will be $\{N \times (N-1)\}/2$ equilibrium equations for goal-replicating assets.
(iii) For each I, there will be N equalities for Z.
(iv) For each I, there will be $N-1$ equations that express the equilibrium between the X and Y variables as in Equations (22) and (23).

5 Shortcomings and Extensions

In this section, we consider some extensions or shortcomings to the model because GRAPM was developed using very simple assumptions.

5.1 Extensions

The simple GRAPM modeled here ignores the weights of each type of investor in the economy and hence is independent of the weight terms.[28] In effect, each goal can be seen as a different class of investor. Permitting free trade in a single market ensures that asset prices are determined by the interaction of these different classes of investors, but weighted by the proportion of individuals/assets in each goal. If we include this assumption, Zeta becomes a function of the weights of each goal based on the assets allocated to the goal (say w_{G1} and w_{G2}, respectively); more explicitly, $X_{I,G1,G2}$ and $Y_{I,G1,G2}$ in Equations (10) and (11) include w_{G1} and w_{G2}, but the allocation formulas are unchanged.[29]

The next simple extension is to assume that the agent's portfolio, I, is made up of multiple assets as opposed to a single risky (or alternatively, that an investor hires multiple agents to diversify the risk of being wrong). This just makes the model a bit more complicated. Because the correlation of a portfolio of assets to a goal-specific asset is nothing more than a weighted sum of each asset correlation to the goal-specific asset, the more complex model can be solved using M-cube risk-adjusted performance. Muralidhar (2001) demonstrates how this is achieved (in the context of hiring multiple agents) and further how assets that might have been considered valuable from a diversification perspective in an absolute return–risk world, may be sub-optimal in a relative risk world (and vice versa).[30] If M is the portfolio of risky assets, such that

$$r(M) = \sum_j w_j r(j), \quad \text{then}$$

$$\rho(M, G1) = \frac{\sum_j w_j \rho(j, G1)\sigma(j)}{\sigma(M)}. \qquad (24)$$

In this multi-asset (or multi-agent) setting, the allocation to each risky asset j that is in this portfolio $M = a * w_j$ and can be solved iteratively (where "a" is the allocation to the risky asset from the M-cube solution). This potentially leads to an interesting new research avenue as to how the optimal (multi-asset) risky portfolio in $G1$ relates to the optimal risky portfolio in $G2$. This offers a new twist to the notion of diversification and also has some ability to explain why many pension funds experienced declines in funded status during the equity crises of 2000-2 and 2008. In short, these pensions used mean–variance optimization (in an absolute return–risk space) to establish optimal portfolios (implicitly assuming as in MPT/CAPM that the goal is deterministic), while ignoring the fact that they had a stochastic goal that essentially resembled a long-duration bond asset. Hence, assets that looked attractive from a diversification perspective in MPT (e.g., equities) proved to be highly risky and even negatively correlated to the goal during crisis events. These assets might not have been included in a GRAPM portfolio for a limited relative risk budget. Coqueret *et al*. (2017) attempt to address this challenge of capturing the goal-hedging value of specific equities as opposed to holding a "market portfolio" proxy. One can also potentially see this from Figure 1.

Third, it is possible to have different relative risk budgets for different goals as the asset pricing equations are independent of $\rho(\tau, L)$, while the asset allocations are impacted by this variable.

5.2 *Shortcomings*

GRAPM does provide additional equations to ensure that there are pair-wise equilibria, but there is no guarantee that these can be forecast or tested empirically. First, it is a normative model and the majority of investors currently do not use M-cube to structure portfolios. However, this paper seeks to address that issue by showing how maximizing M-cube may be the way to invest assets. Second, the multiple equilibrium equations, especially as shown when additional assets are added in Section 4, could make this very complex. Third, it appears that the equilibrium does not just require expected returns to have a precise value, but rather the correlations and volatilities too. GRAPM, like CAPM or RAPM, also requires forecasts of expected returns of the goal-specific relative risk-free assets and hence is subject to the same challenge raised by Housel (2005); namely, our inability to forecast these variables. This might make the model quite difficult to evaluate as the number of assets in the market is rather large, but with big data capabilities and meaningful computational power readily available, this may be computationally feasible and will be explored in future research. Fourth, it requires detailed input on the proportion of each type of investor (w_{Gj}), and their respective goal-replicating asset, and this data is not readily available. Fifth, currently assets do not exist for all goals; while Brazil has created instruments for retirement investing for a child's college,[31] this is not true of the rest of the world. However, this is also an opportunity for innovation. It will be interesting to see if GRAPM can deliver accurate expected returns estimates of such a security.

6 Conclusion

Goals-based investing is slowly becoming the norm for investors, and investors have multiple stochastic goals, delegate to agents (that they hope are skillful), budget risk in a precise manner, and seek to maximize goal-relative risk-adjusted returns. GRAPM incorporates these four (key) positive aspects of investment reality in a single, simple model that captures the relative aspects of investing and provides investors with the three consistent and robust facets of investing—asset pricing equations, asset allocation recommendations, and risk-adjusted performance measures.

From a purely theoretical perspective, this paper demonstrates that, with these positive observations about investor behavior, two- goal-replicating assets and the traditional risk-free asset, all risky assets can be priced. With Brazil issuing two such bonds in January 2023 and August 2023, the theory becomes a reality. The pair-wise asset pricing model is derived from the simple idea that a relative risk-free asset for one goal is a risky asset for another. With free trading of assets, these two assets, plus the absolute risk-free rate, allow us to triangulate to establish the returns for all assets, based on two normalizations of volatility and correlations. Two optimal portfolios for the same goal, with identical volatilities and correlations to the goal, cannot have different returns, and this facet drives the model. This optimal portfolio construction on the part of investors creates a "pair-wise equilibrium." Adding more assets just creates a lattice of "pair-wise" equilibria, more equilibrium equations that must be satisfied, and potentially limits "free parameters". Moreover, in equilibrium, the return of the absolute risk-free asset has an explicit value, and is not exogenous. Furthermore, the asset pricing equations for goal-replicating assets are different from the equations for risky assets, and the asset pricing equation for risky assets incorporates the dual role of an asset in serving as a hedge for a goal and/or a risky asset for another goal. This approach also lends itself easily to an asset pricing model with heterogeneous investors. It also offers new avenues for research as this approach is easily extended by relaxing some of the assumptions made to arrive at this model.

From a practical perspective, this paper provides an optimal method to construct portfolios using agents, to maximize the risk-adjusted performance of the portfolio for each single stochastic goal, for a given explicit risk budget, and maximize the likelihood of hiring/paying agents based on their skill. GRAPM could be used to explain why, in practice, investors have different expected return forecasts (if their forecast is based solely on their goal and not on the global equilibrium)—a very common occurrence in the asset management industry. Furthermore, forecasts of expected returns have been poor both in terms of level and direction and hence potentially having a range of forecasts may be more valuable for practitioners.[32] Furthermore, GRAPM also offers an explanation as to why/how two investors with identical specifications of absolute risk and tracking error can have vastly different asset allocations, because their goals/goal-replicating portfolios and/or their definition of "risky asset" may differ. Most importantly, GRAPM thus has the potential to benefit investors seeking to achieve multiple, stochastic goals (and not just a single goal) that they have to contend with, with limited risk budgets that they seek to optimally exploit, by providing more tailored investment recommendations.

As Adam Smith notes, "He is led by an invisible hand to promote an end which was no part his intention."[33] This paper has shown how individuals maximizing their goal-specific risk-adjusted return, and likelihood of hiring skillful agents, for a given risk budget and goal, could lead to a very compact market equilibrium and pricing model with interesting theoretical and practical implications.

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Appendix I Luck versus Skill Formula

Outperformance over a benchmark does not tell the investor whether the agent is skillful, even if it is normalized by the relative risk taken to achieve this outperformance (i.e., the Differential Sharpe Ratio). Nor does it provide the investor with a measure of confidence that excess returns were generated by skill-based processes as performance data is noisy. Critical factors involved in answering the luck versus skill question include time, the desired degree of confidence, the investment returns of the portfolio and the benchmark, the standard deviation of the portfolio and the benchmark, and the degree of correlation between the two. The more volatile the portfolio and an agent's excess return series, the greater the noise and, hence, the more the time needed to resolve the issue.

Ambarish and Seigel (1996) frame the question in a truly dynamic world and derive useful equations which can inform investors about the skill of a manager.

Assume that portfolio (P) and the benchmark (L) are the two portfolios under consideration.

In finance it is typical to think of risk as variance or the squared standard deviation of returns from the mean return. Therefore, in a relative comparison, one would want to define the variance of the relative return per unit of time as the difference of the portfolio and the benchmark. In other words, one would want the following to hold:

$$\sigma_R^2 = \text{variance per unit of time of } \left(\frac{dP}{P} - \frac{dL}{L}\right) \tag{A.1}$$

follow the generalized Weiner process so that

$$\frac{dP}{P} = \mu_P dt + \sigma_P dz \tag{A.2}$$

$$\frac{dL}{L} = \mu_L dt + \sigma_L dz \tag{A.3}$$

where $(\mu_P,\ \sigma_P)$ and $(\mu_L,\ \sigma_L)$ are the instantaneous mean and volatility parameters of the portfolio and benchmark, respectively. Define $\rho_{P,L}$ as the correlation between P and I and define the ratio of the portfolio and benchmark as $R(t)$ such that

$$R(t) = \frac{P(t)}{L(t)}. \tag{A.4}$$

The dynamics of $R(t)$ can be extracted using Ito's Lemma such that

$$\frac{dR}{R} = (\mu_p - \mu_L + \sigma_L^2 - \sigma_P \sigma_L \rho_{P,L})dt + \sigma_P dz_p - \sigma_L dz_L. \quad \text{(A.5)}$$

As noted in the original article, "the variance of the stochastic terms in Equation (A.5) are identical to those in Equation (A.1), completing the intuitive appeal of using $R(t)$ as measure of relative performance."

Since we are interested in the time development of $R(t)$, define dz such that

$$\sigma_R dz = \sigma_P dz_p - \sigma_L dz_L. \quad \text{(A.6)}$$

Squaring both sides and reducing since $dz^2 = \text{dt}$ and $dz_P * dz_L = \rho_{P,L}\text{dt}$

$$\sigma_R^2 = \sigma_P^2 + \sigma_L^2 - 2 * \sigma_P * \sigma_L * \rho_{P,L}. \quad \text{(A.7)}$$

Equation (A.7) is very interesting as it is exactly the square of the tracking error in Equation (7), and the square of the denominator in the Differential Sharpe Ratio. This validates the use of this formulation for the relative risk measure that Sharpe (1994) used.

One can go a step further and derive a simple geometric Brownian motion from Equation (A.5) for R such that

$$R(t) = R(0) * e^{\sigma_R \epsilon \sqrt{t}} * e^{\left[\left\{\left(\mu_P - \frac{\sigma_P^2}{2}\right) - \left(\mu_L - \frac{\sigma_L^2}{2}\right)\right\}t\right]} \quad \text{(A.8)}$$

where $\in$ is the standard normal variable. The first exponential factor in Equation (A.8) is the noise component and the second exponent is the true skill in added value. Therefore, for skill to dominate noise, a minimum number of data points are needed or alternatively,

$$T > \frac{K^2 * (\sigma_P^2 - \sigma_L^2 - 2 * \sigma_P * \sigma_L * \rho_{P,L})}{\left\{\left(\mu_P - \frac{\sigma_P^2}{2}\right) - \left(\mu_L - \frac{\sigma_L^2}{2}\right)\right\}^2} \quad \text{(A.9a)}$$

where K is the number of standard deviations for a given confidence interval. For example, when K = 1, then one desires an 84% confidence in the skill of the manager.

Through cross multiplication, and taking the square root of both sides, results

$$\sqrt{T} * \left[\frac{\left\{\left(\mu_P - \frac{\sigma_P^2}{2}\right) - \left(\mu_L - \frac{\sigma_L^2}{2}\right)\right\}}{\sqrt{(\sigma_P^2 - \sigma_L^2 - 2 * \sigma_P * \sigma_L * \rho_{P,L})}}\right] > K. \quad \text{(A.9b)}$$

For each agent, one can input the values on the left-hand side of Equation (A.9b) and deduce the confidence in skill. Since Equation (A.9b) isolates the differences in volatility (between the agent's portfolio and benchmark) in the numerator, and normalized for TE in the denominator, it is easy to see why this approach rates agents identical to M-cube. M-cube normalizes for differences in volatility and correlation (i.e., together TE).

Appendix II Derivation of M-Cube Risk-Adjusted Performance

Principals will

$$\max E[r(A) - r(L)] \text{ or Equation (1)} \quad \text{(A.10)}$$

subject to

$$E[r(A)] = a_{\tau/L}^P \times E[r(P)] + l_{\tau/L}^L \times E[r(L)] + (1 - a_{\tau/L}^P - l_{\tau/L}^L) \times r(F) \quad \text{(A.11)}$$

$$\sigma(A) = \sigma(L) \quad \text{(A.12)}$$

and

$$\text{TE}(A, L) = \text{TE}(\text{Target}, L) \quad \text{(A.13)}$$

where $a^P_{\tau/L}$ is the allocation to the risky (agent) portfolio, P, given the goal of L and a target relative risk of τ; $l^L_{\tau/L}$ is the allocation to the goal-hedging/goal-replicating asset, L (and measures what is *de facto* invested in the low-cost passive benchmark or "beta" in industry parlance). The superscript denotes the asset being allocated to; the subscript indicates the target relative risk (τ) and the goal-replicating asset (L). The balance of the assets is invested in the traditional absolute risk-free asset, F (and measures leverage).

Now

$$\mathrm{TE}(A, L) = \sqrt{\begin{array}{l}[\sigma^2(A) - 2 * \rho(A, L) \\ \quad * \sigma(A) * \sigma(L) + \sigma^2(L)]\end{array}} \tag{A.14}$$

where $\rho(*)$ is the correlation parameter. From the constraint on tracking error (Equation (6) or (A.13)), a unique target correlation between portfolio A and liability L, $\rho(\tau, L)$, is identified. Alternatively, instead of specifying a TE(Target, L), a principal could also specify $\rho(\tau, L)$.

Substituting Equation (A.14) into Equation (A.13) and squaring both sides, and substituting for $\sigma(A) = \sigma(L)$ obtains

$$\begin{aligned}\rho(TL) &= 1 - \frac{\mathrm{TE}(\mathrm{Target}, L)^2}{2 \times \sigma^2(L)} \\ &= 1 - \frac{\tau^2}{2 \times \sigma^2(L)}.\end{aligned} \tag{A.15}$$

The optimal allocations to "a" and "l" ensures that $\rho(A, L)$ equals $\rho(\tau, L)$ and $\sigma(A) = \sigma(L)$.

Further, calculating $\sigma^2(L) = \sigma^2(A)$ using Equations (A.11) and (A.12) gives

$$\begin{aligned}\sigma^2(L) = (a^P_{\tau/L})^2 \times \sigma^2(P) + (l^L_{\tau/L})^2 \\ \times \sigma^2(L) + 2 \times (a^P_{\tau/L}) \times (l^L_{\tau/L}) \\ \times \sigma(P) \times \sigma(L) \times \rho(P, L).\end{aligned} \tag{A.16}$$

But, the covariance of A and L can be expressed below using the definition of covariance and Equation (A.11) as

$$\begin{aligned}\sigma(A) \times \sigma(L) \times \rho(A, L) \\ = a^P_{\tau/L} \times \sigma(P) \times \sigma(L) \times \rho(P, L) \\ + l^L_{\tau/L} \times \sigma^2(L).\end{aligned} \tag{A.17}$$

Using Equation (A.12), and imposing $\rho(A, L) = \rho(\tau, L)$ implied from Equation (A.13) as an optimal condition, Equation (A.17) can be rewritten as

$$\begin{aligned}\rho(\tau, L) \times \sigma^2(L) = a^P_{\tau/L} \times \sigma(P) \times \sigma(L) \\ \times \rho(P, L) + l^L_{\tau/L} \times \sigma^2(L).\end{aligned} \tag{A.18}$$

Dividing the RHS and LHS by $\sigma^2(L)$ and rearranging terms,

$$l^L_{\tau/L} = \rho(\tau, L) - a^P_{\tau/L} \times \frac{\sigma(P)}{\sigma(L)} \times \rho(P, L). \tag{A.19}$$

Substituting Equation (A.19) into Equation (A.18) and solving gives

$$\begin{aligned}a^P_{\tau/L} &= \frac{\sigma(L)}{\sigma(P)} \left[\sqrt{\frac{[1 - \rho(\tau, L)^2]}{[1 - \rho(P, L)^2]}} \right] \\ &= \frac{\sigma(L)}{\sigma(P)} \left[\frac{\varphi(\tau, L)}{\varphi(P, L)} \right],\end{aligned}$$

where

$$\varphi(I, J) = \sqrt{[1 - \rho(I, J)^2]} \tag{A.20}$$

And substituting Equation (A.19) into Equation (A.20) gives

$$\begin{aligned}l^L_{\tau/L} &= \rho(\tau, L) - \rho(P, L) \\ &\quad \times \left[\sqrt{\frac{[1 - \rho(\tau, L)^2]}{[1 - \rho(P, L)^2]}} \right] \\ &= \rho(\tau L) \rho(P, L) \times \left[\frac{\varphi(\tau, L)}{\varphi(P, L)} \right].\end{aligned} \tag{A.21}$$

Appendix III Applying the M-Cube Model Results for Asset Allocation to the City of Austin's Employees' Retirement System (January 2020)[34]

Assume $\sigma(L) = 10\%$, $\sigma(P) = 10\%$, and $\rho(P, L) = 0.98$. Then a TE(Target) $= 1.5\%$, implies that $\rho(\tau, L) \approx 0.9887$. Figure A.1 provides a range of tracking error targets around this TE(Target) by just varying $\rho(\tau, L)$. The allocation to these broad group of assets for 1.5% target tracking error coincidentally aligned with the actual target allocation.

Appendix IV Eightfold Path to Deriving GRAPM

Define $E[r(A)_{x,y|\tau}]$ to be the expected risk-adjusted return of portfolio A with x as the risky asset and y as the goal, with τ as the target relative risk. Recall Equations (A.20) and (A.21) establish optimal allocations to risky, goal-replicating, and absolute risk-free (F) assets for investors maximizing goal-relative expected risk-adjusted returns, given target absolute and relative risk levels. At this stage, this is an optimal, but a non-equilibrium result.

(1) First consider goal $G1$ in isolation. Use Equation (4) to establish the expected risk-adjusted return of a portfolio for the $G1$ goal with I as risky asset, and goal-replicating asset $G1$ and define it as $E[r(A)_{I,G1|\tau}]$. This is shown in Equation (A.22) which is a repeat of the M-cube formulation.

$$E[r(A)_{I,G1|\tau}] = \left\{\frac{\sigma(G1)}{\sigma(I)}\left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right\} \times E[r(I)] + \left\{\rho(\tau, G1) - \rho(I, G1) \times \left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right\} \times E[r(G1)]$$

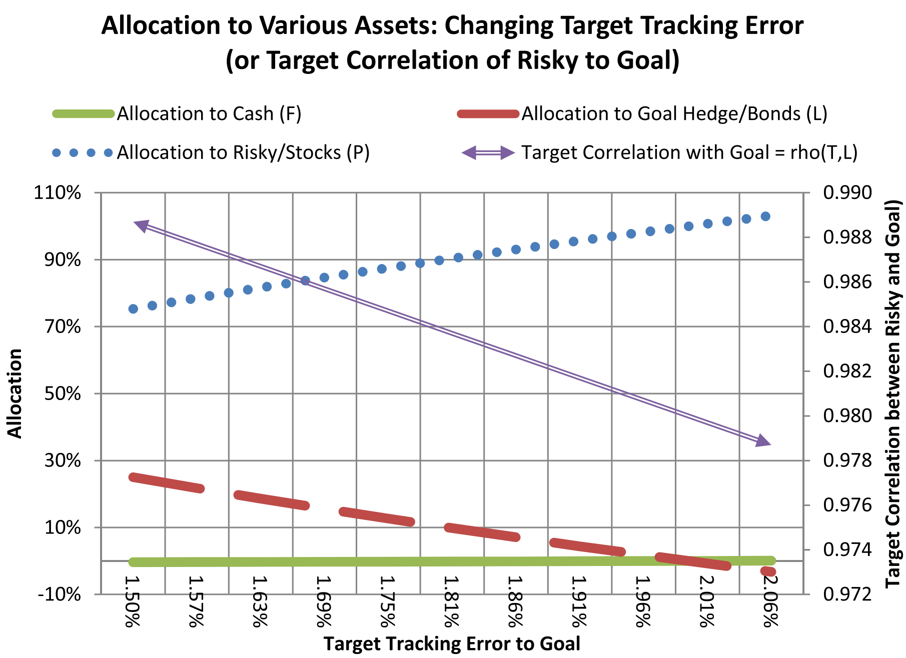

Figure A.1 Allocation to risky assets, goal-replicating asset, and absolute risk-free asset for different target tracking errors.

$$+\left(1-\frac{\sigma(G1)}{\sigma(I)}\left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right.$$
$$-\left\{\rho(\tau, G1)-\rho(I, G1)\right.$$
$$\left.\left.\times\left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right\}\right)\times r(F). \quad \text{(A.22)}$$

(2) Similarly, establish the expected risk-adjusted return of a portfolio for the $G1$ goal with $G2$ as risky asset and define it as $E[r(A)_{G2,G1|\tau}]$. This is shown in Equation (A.23).

$$E[r(A)_{G2,G1|\tau}]$$
$$=\left\{\frac{\sigma(G1)}{\sigma(G2)}\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right\}\times E[r(G2)]$$
$$+\left\{\rho(\tau, G1)-\rho(G2, G1)\right.$$
$$\left.\times\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right\}\times E[r(G1)]$$
$$+\left(1-\frac{\sigma(G1)}{\sigma(G2)}\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right.$$
$$-\left\{\rho(\tau, G1)-\rho(G2, G1)\right.$$
$$\left.\left.\times\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right\}\right)\times r(F). \quad \text{(A.23)}$$

(3) This is the most important step and statement in this paper. If a risk-adjusted portfolio has the same volatility as the goal ($G1$) and the same target correlation ($\rho(\tau, G1)$), then it cannot have two expected values based on which asset was used to create it (unless there is an arbitrage opportunity or a market imperfection). In other words, the equilibrium condition requires that $E[r(A)_{I,G1|\tau}] = E[r(A)_{G2,G1|\tau}]$.[35] This equality between Equations (A.22) and (A.23) allows us to derive $E[r(I)_{G1}]$ as a function of $G1$, $G2$, and F as in Equation (A.27). For convenience, we will dispense with τ in the subscript as the terms below are independent of τ.

If

$$a'_{G1}=\frac{\sigma(I)}{\sigma(G2)}\left[\frac{\varphi(I, G1)}{\varphi(G2, G1)}\right]. \quad \text{(A.24)}$$

And

$$l'_{G1}=\frac{\sigma(I)}{\sigma(G1)}\times\rho(I, G1)-\frac{\sigma(I)}{\sigma(G1)}$$
$$\times\rho(G2, G1)\times\left[\frac{\varphi(I, G1)}{\varphi(G2, G1)}\right]. \quad \text{(A.25)}$$

Define $E[r(I)_{G1}]$ as the expected return of asset I in $G1$. Then, setting Equation (A.22) = Equation (A.23) gives

$$E[r(I)_{G1}]=a'_{G1}\times E[r(G2)]$$
$$+\,l'_{G1}\times E[r(G1)]$$
$$+\,(1-a'_{G1}-l'_{G1})\times r(F). \quad \text{(A.26)}$$

Alternatively, Equation (A.26) can be expressed in the following manner by re-arranging terms.

$$E[r(I)_{G1}-r(F)]$$
$$=a'_{G1}\times E[r(G2)-r(F)]$$
$$+\,l'_{G1}\times E[r(G1)-r(F)]. \quad \text{(A.27)}$$

It may appear that Equation (A.27) offers the asset pricing equation desired, but this is only a partial result because it is based entirely on just one set of investors—those with $G1$ as the goal or the myopic perspective. So far, there is no limit on the parameters. It is the heterogeneity of investors/goals that gives the full equilibrium and ensures no "free parameters".

(4) Similarly, for the $G2$ goal, follow Steps 1–3 to derive $E[r(A)_{I,G2}]$ and $E[r(A)_{G1,G2}]$. Once again, as in Step 4, set $E[r(A)_{I,G2}] = E[r(A)_{G1,G2}]$ to derive $E[r(I)_{G2}]$ as function of $G1$, $G2$, and F as in Equation (A.31).

If

$$a''_{G2} = \frac{\sigma(I)}{\sigma(G1)}\left[\frac{\varphi(I,G2)}{\varphi(G1,G2)}\right]. \quad \text{(A.28)}$$

And

$$l''_{G2} = \frac{\sigma(I)}{\sigma(G2)} \times \rho(I,G2) - \frac{\sigma(I)}{\sigma(G2)} \times \rho(G1,G2) \times \left[\frac{\varphi(I,G2)}{\varphi(G1,G2)}\right]. \quad \text{(A.29)}$$

Then

$$E[r(I)_{G2}] = a''_{G2} \times E[r(G1)] + l''_{G2} \times E[r(G2)] + (1 - a''_{G2} - l''_{G2}) \times r(F). \quad \text{(A.30)}$$

Or alternatively, re-arranging terms in Equation (A.30),

$$E[r(I)_{G2} - r(F)] = a''_{G2} \times E[r(G1) - r(F)] + l''_{G2} \times E[r(G2) - r(F)]. \quad \text{(A.31)}$$

(5) Now assume the representative investors across the two goals can trade freely. We assume that asset I cannot have two different returns for investors with goal $G1$ and goal $G2$, unless there is a market inefficiency. In other words, an equilibrium between the two sets of representative investors requires that $E[r(I)_{G1}] = E[r(I)_{G2}]$ (or Equation (A.27) = Equation (A.31)).

Recall that there are three classes of assets and GRAPM provides specific expected returns for each asset. This condition gives two key equilibrium conditions: Equation (A.34) for $r(F)$ and Equation (A.35) for the relationship between goal-replicating assets, $G1$ and $G2$. If we define

$$X_{I,G1,G2} = \varphi(I,G1) - \rho(I,G2) \times \varphi(I,G1) + \rho(G1,G2) \times \varphi(I,G2) \quad \text{(A.32)}$$

$$Y_{I,G1,G2} = \varphi(I,G2) - \rho(I,G1) \times \varphi(I,G2) + \rho(G1,G2) \times \varphi(I,G1). \quad \text{(A.33)}$$

Then

$$r(F)_{G1,G2} = \frac{\left\{E[r(G1) - \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times E[r(G2)]\right\}}{\left[1 - \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right)\right]}. \quad \text{(A.34)}$$

Alternatively, Equation (A.27) can be restated as

$$E[r(G1) - r(F)] = \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times E[r(G2) - r(F)]. \quad \text{(A.35)}$$

Using Equations (A.28), (A.29), and (A.35) in Equation (A.31) yields the key asset pricing Equation (A.36) for risky asset I, using $G1$ as a risky asset and $G2$ as the goal.

$$E[r(I) - r(F)] = Z_{I,G1/G2} \times E[r(G1) - r(F)]. \quad \text{(A.36)}$$

Where Zeta is a form of a "covariance term" as beta is in CAPM, defined as

$$Z_{I,G1/G2} = \frac{\sigma(I)}{\sigma(G1)} \times \left\{ \left[\frac{\varphi(I, G2)}{\varphi(G1, G2)} \right] + \left(\frac{Y_{I,G1,G2}}{X_{I,G1,G2}} \right) \times \rho(I, G2) - \left(\frac{Y_{I,G1,G2}}{X_{I,G1,G2}} \right) \times \rho(G1, G2) \times \left[\frac{\varphi(I, G2)}{\varphi(G1, G2)} \right] \right\}. \quad \text{(A.37)}$$

Notice, that Zeta is a function of two volatilities, $\sigma(I)$ and $\sigma(G1)$ (as in CAPM), but three correlations: $\rho(G1, G2)$, $\rho(I, G2)$, and $\rho(I, G1)$ One other key point to note, Equation (A.36) cannot be used for assets that are goal-replicating; instead, Equation (A.35) must be used for goal-replicating assets. Furthermore, since X and Y are functions of $\varphi(I, J)$, Zeta is not defined when $\rho(G1, G2)$, $\rho(I, G2)$ or $\rho(I, G1) = 1$, unless the other correlations are = 0.

Interestingly, Zeta can be restated in terms of "a" and "l" using Equations (13) and (14) as follows and will be interpreted later:

$$Z_{I,G1/G2} = \frac{a^{G1}_{\tau/G2}}{a^{I}_{\tau/G2}} + \frac{\sigma(I)}{\sigma(G1)} \times \left(\frac{Y_{I,G1,G2}}{X_{I,G1,G2}} \right) \times l^{G2}_{I/G2} \quad \text{(A.38)}$$

(6) Similarly, express $E[r(I)]$ as a function of $E[r(G2)]$, $r(F)$, volatilities and correlations (assuming $G1$ is the goal) as in Equation (A.39).

$$E[r(I) - r(F)] = Z_{I,G2/G1} \times E[r(G2) - r(F)] \quad \text{(A.39)}$$

In a way, we can see these dual Equations in (A.36) and (A.39) for I as a pair-wise or "Janus" equilibrium.[36]

(7) Interestingly, given Equations (A.39), (A.35), and (A.36), there is a unique relationship between $Z_{I,G1/G2}$ and $Z_{I,G2/G1}$; namely, that

$$Z_{I,G2/G1} = \frac{\sigma(G1)}{\sigma(G2)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}} \right) \times Z_{I,G1/G2}. \quad \text{(A.40)}$$

Appendix V The Three Goal-Replicating GRAPM

With three goals: namely, $G1$, $G2$ and $G3$, there are now six potential equations for the expected returns of asset I. In equilibrium, all must provide the same result because all assets are linked by the "pair-wise" equilibrium.

$$E[r(I) - r(F)] = Z_{I,G3/G1} \times E[r(G3) - r(F)] \quad \text{(A.41)}$$

$$E[r(I) - r(F)] = Z_{I,G3/G2} \times E[r(G3) - r(F)] \quad \text{(A.42)}$$

$$E[r(I) - r(F)] = Z_{I,G1/G3} \times E[r(G1) - r(F)] \quad \text{(A.43)}$$

$$E[r(I) - r(F)] = Z_{I,G2/G3} \times E[r(G2) - r(F)]. \quad \text{(A.44)}$$

Interestingly, given Equations (A.39) and (A.44) and the fact that Zeta is applied to the same risk premium, $E[r(G2) - r(F)]$, and continuing in a similar fashion for all other goal-replicating assets (in this case $E[r(G1) - r(F)]$ and $E[r(G3) - r(F)]$), then this must be the case that

$$Z_{I,G2/G1} = Z_{I,G2/G3}. \quad \text{(A.45)}$$

From Equations, (A.41) and (A.42), we can derive Equation (A.46).

$$Z_{I,G3/G1} = Z_{I,G3/G2}. \quad \text{(A.46)}$$

And from Equations (A.36) and (A.43), we can derive Equation (A.47).

$$Z_{I,G1/G2} = Z_{I,G1/G3}. \tag{A.47}$$

Notice that Equations (A.45), (A.46), and (A.47) are a bit different from Equation (A.40) because in these equations the second goal-replicating asset is identical on both sides of the equation (e.g., $G2$ in Equation (A.45)), but not so in Equation (A.40). Using the approach to derive Equation (A.40) could provide additional relationships among the Zetas, given the three-goal case.

This "Janus" duality of Zeta is interesting and in contrast to the single "beta" one derives from CAPM. Furthermore, given the equilibrium condition in Equation (A.35) between two goal replicating assets, we can additionally show that,

$$E[r(G1) - r(F)] = \frac{\sigma(G1)}{\sigma(G3)} \times \left(\frac{X_{I,G1,G3}}{Y_{I,G1,G3}}\right) \times E[r(G3) - r(F)] \tag{A.48}$$

$$E[r(G2) - r(F)] = \frac{\sigma(G2)}{\sigma(G3)} \times \left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right) \times E[r(G3) - r(F)]. \tag{A.49}$$

Using Equation (A.49) in Equation (A.35), we can eliminate $E[r(G2) - r(F)]$ and derive Equation (A.50).

$$E[r(G1) - r(F)] = \frac{\sigma(G1)}{\sigma(G3)} \times \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times \left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right) \times E[r(G3) - r(F)]. \tag{A.50}$$

Setting Equation (A.50) = Equation (A.43) gives an additional equilibrium equation; namely, that

$$\left(\frac{X_{I,G1,G3}}{Y_{I,G1,G3}}\right) = \left(\frac{X_{I,G1,G2}}{Y_{I,G1,G2}}\right) \times \left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right). \tag{A.51}$$

From these equations we can also show that

$$\left(\frac{X_{I,G2,G3}}{Y_{I,G2,G3}}\right) = \left(\frac{Y_{I,G3,G2}}{X_{I,G3,G2}}\right). \tag{A.52}$$

And similarly for the other combination of $G1$, $G2$, and $G3$.

Appendix VI Proof of Key Assumption in the Paper

In this Appendix we provide three proofs for why the key assumption in the paper is acceptable; namely, that

$$E[r(A)_{I,G1|\tau}] = E[r(A)_{G2,G1|\tau}]. \tag{A.53}$$

This Appendix addresses this assumption through three methods: (a) an Intuitive Argument; (b) A Specific Case; and (c) the General Result.

(a) *An Intuitive Argument*

If $E[r(A)_{I,G1|\tau}] < E[r(A)_{G2,G1|\tau}]$, then one could argue that no principal would hire the agent that has created a portfolio with asset I, and hence with zero demand, $E[r(I)] = 0$.

Also, the equality can hold with I and $G2$ being completely different assets. Note that $a^{I}_{\tau/G1} \neq a^{G2}_{\tau/G1}$ and similarly, for the allocation to $G1$ in both risk-adjusted portfolios, because I and $G2$ are assumed to be distinct assets.

(b) *A Specific Case*

One may try to argue that $E[r(A)_{I,G1}] = E[r(A)_{G2,G1}]$ is not guaranteed when the two portfolios have the same absolute and relative risk, even with an identical correlation to the goal. One possibility suggested by a reader is that if $r(I)$, $r(G2)$, $r(G1)$ have the same variance, but are independent, then this may not be true. However, if we plug in $\sigma(I) = \sigma(G2) = \sigma(G1)$ and if $\rho(G2, G1) = \rho(I, G1) = \rho(G2, I) = 0$ into our Equation (A.53), then it results in $E[r(I)] = E[r(G2)]$, which violates our assumption that the risky assets are unique. As a result, we must rule out this special case.[37]

(c) *The General Result*

The third option starts with the expanded form of both sides of Equation (A.53). From Equations (A.22) and (A.23), assume that Equation (A.53) is not true. Then,

$$\begin{aligned}
&\left\{\frac{\sigma(G1)}{\sigma(I)}\left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right. \times E[r(I)] \\
&\quad + \left\{\rho(\tau, G1) - \rho(I, G1) \times \left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right\} \\
&\quad \times E[r(G1)] + \left(1 - \frac{\sigma(G1)}{\sigma(I)}\left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right. \\
&\quad - \left\{\rho(\tau, G1) - \rho(I, G1)\right. \\
&\quad \left.\left. \times \left[\frac{\varphi(\tau, G1)}{\varphi(I, G1)}\right]\right\}\right) \times r(F) \\
&\neq \left\{\frac{\sigma(G1)}{\sigma(G2)}\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right. \times E[r(G2)] \\
&\quad + \{\rho(\tau, Gl) - \rho(G2, G1) \\
&\quad \left. \times \left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right\} \times E[r(G1)] \\
&+ \left(1 - \frac{\sigma(G1)}{\sigma(G2)}\left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right. \\
&\quad - \{\rho(\tau, G1) - \rho(G2, G1) \\
&\quad \left.\left. \times \left[\frac{\varphi(\tau, G1)}{\varphi(G2, G1)}\right]\right\}\right) \times r(F). \qquad \text{(A.54)}
\end{aligned}$$

Eliminating common terms from both the LHS and RHS.

$$\begin{aligned}
&\left\{\frac{\sigma(G1)}{\sigma(I)}\left[\frac{1}{\varphi(I, G1)}\right]\right\} \times E[r(I)] \\
&\quad + \left\{-\rho(I, G1) \times \left[\frac{1}{\varphi(I, G1)}\right]\right\} \\
&\quad \times E[r(G1)] \\
&\quad + \left\{-\frac{\sigma(G1)}{\sigma(I)}\left[\frac{1}{\varphi(I, G1)}\right] \times r(F)\right\} \\
&\quad - \left\{-\rho(I, G1) \times \left[\frac{1}{\varphi(I, G1)}\right]\right\} \times r(F) \\
&\neq \left\{\frac{\sigma(G1)}{\sigma(G2)}\left[\frac{1}{\varphi(G2, G1)}\right]\right\} \times E[r(G2)] \\
&\quad + \left\{-\rho(G2, G1) \times \left[\frac{1}{\varphi(G2, G1)}\right]\right\} \\
&\quad \times E[r(G1)] \\
&\quad + \left\{-\frac{\sigma(G1)}{\sigma(G2)}\left[\frac{\varphi(T, G1)}{\varphi(G2, G1)}\right]\right\} \times r(F) \\
&\quad - \left\{-\rho(G2, G1) \times \left[\frac{\varphi(T, G1)}{\varphi(G2, G1)}\right]\right\} \\
&\quad \times r(F) \qquad \text{(A.55)}
\end{aligned}$$

$$\begin{aligned}
&\left\{\frac{\sigma(G1)}{\sigma(I)}\left[\frac{1}{\varphi(I, G1)}\right]\right\} \\
&\quad \times E[r(I) - r(F)] \\
&\quad - \left\{\rho(I, G1) \times \left[\frac{1}{\varphi(I, G1)}\right]\right\} \\
&\quad \times E[r(G1) - r(F)] \\
&\neq \left\{-\rho(G2, G1) \times \left[\frac{1}{\varphi(G2, G1)}\right]\right\} \\
&\quad \times E[r(G1) - r(F)]. \qquad \text{(A.56)}
\end{aligned}$$

Which boils down to Equation (A.57), which we could add as an additional assumption to the model to ensure that the model is robust.

$$\sigma(G1)\left[\frac{E[r(I)-r(F)]}{\sigma(I)\varphi(I,G1)}-\frac{E[r(G2)-r(F)]}{\sigma(G2)\varphi(G2,G1)}\right] \neq \left[\frac{\rho(G2,G1)}{\varphi(G2,G1)}-\frac{\rho(I,G1)}{\varphi(I,G1)}\right] \times E[r(G1)-r(F)]. \quad (A.57)$$

Using the assumptions in Appendix VI.b, this results in $E[r(I)] \neq E[r(G2)]$.

Endnotes

1 https://www.brainyquote.com/quotes/quotes/c/confucius140548.html?src=t_goals

2 See for example the statement of objectives of the New Mexico Public Employees' Retirement Association. http://www.nmpera.org/assets/uploads/downloads/RIO/RFP/RFP-NO.-NM-INV-001-FY19-Total-Fund-Overlay-Services.pdf

3 Das *et al*. (2018) focus on just optimal asset allocation for multiple stochastic goals; not asset pricing nor risk-adjusted performance.

4 Merton and Muralidhar (2016) call for the creation of a new "safe" asset for retirement, and Muralidhar (2016) argues for the creation of new "safe" assets for retirement and saving for a child's college education. Additionally, Muralidhar (2019b) recommends the creation of LIVE bonds to help individuals and insurance companies hedge longevity risk. One can easily extend these ideas to other goals.

5 https://www.tesourodireto.com.br/data/files/86/87/90/ED/EE85581048AB1358894D49A8/Tesouro%20RendA_.pdf. While this presentation is in Portuguese, the cash flows highlighted on page 14 show how these forward-starting cash flows are different from the cash flows of a traditional bond.

6 https://comptroller.nyc.gov/services/financial-matters/pension/asset-allocation/

7 Thanks to Prof. Robert C. Merton for stimulating this insight.

8 Per the kind suggestion of the reviewer we also demonstrate this in the context of another public pension fund—City of Austin Employees Retirement System (CoAERS) based on analysis conducted in 2019.

9 http://www.nmpera.org/assets/uploads/downloads/RIO/RFP/RFP-NO.-NM-INV-001-FY19-Total-Fund-Overlay-Services.pdf

10 Or alternatively, if they delegate, they do not restrict the relative risk and do not care whether the agent is skillful. This is explained later in Section 2.

11 The similarities to the Modigliani–Miller (1958) approach should be obvious to the reader as this procedure deleverages any uncompensated leverage that an agent might apply.

12 http://www.nmpera.org/assets/uploads/downloads/RIO/RFP/RFP-NO.-NM-INV-001-FY19-Total-Fund-Overlay-Services.pdf

13 https://www.lacera.com/sites/default/files/assets/documents/general/invest_policy_stmt_031921.pdf–page 8.

14 Brennan (1993) and Cornell and Roll (2005) address agency, but ignore the skill considerations.

15 See Equation (7).

16 Which is an exogenous relative risk target value specified by the principal and derived from TE(Target) once $\sigma(L)$ is known.

17 Appendix V explains the basis for this assumption using an intuitive method (based on assumption III.a.iii), a specific case and a general approach. It also discusses potential additional constraints needed for the model to guarantee this result.

18 In just the retirement world, Social Security systems, corporate and public defined benefit pension plans are underfunded and there is evidence that even individuals are not saving enough in their defined contribution plans. Hence, this is a very reasonable assumption.

19 In many pension funds, the board of the pension fund is composed of non-financial individuals who have other jobs. In the retail world, close to 40% of the population is financially unsophisticated.

20 Jorion (2003) argues that normalizing absolute volatility, in exactly this manner, is the efficient way to budget tracking error risk.

21 Implicitly, this assumes that $\rho(G1, G2) \neq 1$ or -1.

22 For simplicity, we are assuming that $\rho(I, G1)$ and $\rho(I, G2) \neq 1$ or -1. This constraint can be relaxed based on other correlations, but for now, it keeps the model simple.

23 In future research, we will detail and examine the impact on these pricing equations when one assumes extreme values for the correlation parameters. It is beyond the scope of this paper.

24 In this example, the Janus duality is demonstrated by the fact that I can be priced in this pair-wise equilibrium with $G1$ as risky ($G2$ as goal) or vice versa. This duality is unchanged even when we add more goal-replicating assets, as we only use two at a time to create the asset pricing equation for I.

[25] A reader has suggested that this could imply a general equilibrium result which will be explored in future research.

[26] One could expand the formula for M to capture the correlation of asset I to all assets in M as shown in Section 5, Equation (24). But this is difficult to comprehend in CAPM as the expanded form is rarely shown or discussed.

[27] This model is simpler than models of investors with heterogeneous expectations/information/curvature in utility functions (Kogan and Lewis, 2000), as those models typically need stylized utility functions (CARA) and assumptions about complete markets.

[28] Thanks to Prof. Robert C. Merton for stimulating this insight.

[29] If w_{G1} is the weight of $G1$ and w_{G2} is the weight of $G2$ (such that the sum is 1), $\mathrm{X}_{I,G1,G2}^{w_{G1},w_{G2}} = w_{G1} \times \varphi(I, G1) - w_{G2} \times \rho(I, G2) \times \varphi(I, G1) + w_{G2} \times \rho(G1, G2) \times \varphi(I, G2)$ and $\mathrm{Y}_{I,G1,G2}^{w_{G1},w_{G2}} = w_{G2} \times \varphi(I, G2) - w_{G1} \times \rho(I, G1) \times \varphi(I, G2) + w_{G1} \times \rho(G1, G2) \times \varphi(I, G1)$.

[30] Muralidhar (2001) goes further to demonstrate how this approach is superior to naïve maximization of differential Sharpe ratios (also known as Information ratios) and other techniques that do not rank multiple agent portfolios consistent with measures of confidence in skill.

[31] https://einvestidor-estadao-com-br.cdn.ampproject.org/c/s/einvestidor.estadao.com.br/investimentos/tesouro-nacional-lancamentos-segundo-semestre/amp

[32] Thanks to Prof. Robert C. Merton for stimulating this insight.

[33] https://www.panmurehouse.org/adam-smith/smith-quotes-faqs/#:∼:text=quotes%20from%20Smith-,'No%20 society%20can%20surely%20be%20flourishing%20and%20happy%2C%20of%20which,'

[34] https://www.coaers.org/

[35] See Appendix V for three approaches to justify this assumption.

[36] In this example, the Janus duality is demonstrated by the fact that I can be priced in this pair-wise equilibrium with $G1$ as risky ($G2$ as goal) or vice versa. This duality is unchanged even when we add more goal-replicating assets, as we only use two at a time to create the asset pricing equation for I.

[37] Thanks to Prof. Kazuhiko Ohashi for this insight.

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