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JOURNAL OF INVESTMENT MANAGEMENT, Vol. 15, No. 3, (2017), pp. 41–52

MULTI-PERIOD PORTFOLIO REBALANCING WITH PERSONAL TAX

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This paper compares two heuristic rebalancing rules for taxable accounts. The first one is trading X percent annually. The second one is based on the result of recent research, which indicate there existence of no-trading zone. The no-trading zone is obtained by using a quadratic function to approximate the optimal value function. We show a simple and implementable approximation. We also show that the no-trading zone-based rebalancing rule performs better than trading X percent annually.

1 Introduction

The practical advice to the multi-period portfolio rebalancing problem with personal tax is to make adjustments over a long period of time to spread out the tax paid. It is easy to understand and implement. We will implement this rebalancing strategy as trading *X* percent of the current portfolio annually. We will show empirically the optimal *X* percent for various starting portfolios and planning periods. The optimal annual trading is small for a well-diversified portfolio and large for a concentrated portfolio.

The modern portfolio theory suggests a very different rebalancing strategy. There exists a zone containing the optimal portfolio. If the current portfolio is within the zone, there is no need to do any adjustment. If the current portfolio is outside the zone, the initial trading should bring the portfolio to the boundary of this zone, and subsequent trading should keep the portfolio at the boundary. We call the above zone no-trading zone. In this paper, we will derive a simple and easily implementable way to solve the no-trading zone analytically. It is based on the assumption that the unknown value function (the maximum achievable utility) can be approximated by a simple quadratic function. The no-trading zone depends on capital gain, risk tolerance, planning period, and the holding differences between the current portfolio and the optimal portfolio. We show empirically that this no-trading zone based rebalancing rule is better than the simply trading *X* percent annually.

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The essence of rebalancing is to find the best trade-off between paying tax now (one time) versus the cumulative future benefits of the optimal portfolio. Since tax is paid only when capital gain is realized, the present value of a one dollar capital gain depends on how long you hold onto it. The popular wisdom is that you should never realize the capital gain unless you have to, and that you should manage your asset exposure by means of a complicated schema, such as shorting or derivative instrument. We don't recommend such a schema for retirees. Constantinides (1984) shows that investors should realize some capital gain in order to increase the chance that the investor can take a capital loss in the future. Realizing capital loss is valuable because the loss can be deducted potentially from income with a higher income tax rate. Constantinides supports his theory empirically by studying the rebalancing effect of holding portfolios over a 15-year period.

Dynamic programming should be the perfect tool for a multi-period rebalancing problem. Markowitz and van Dijk (2003) use it to get optimal rebalancing rule for a simplified problem with a total of 55 possible states. In our more realistic setting with the number of underlying investable assets being 10, the total possible states are astronomically large, which makes dynamic programming impractical even in today's computing environment. When the possible states are continuous and infinite, the dynamic programming principle implies that value function will satisfy the Hamilton–Jacob–Bellman (HJB) equation. This is an nonlinear second-order partial differential equation. The exact value function can be found only for limited cases in the complete market setting. The value function has to be solved numerically for most practical cases.

There is a vast quantity of literature on the subject of portfolio rebalancing in the presence of transaction (trading) costs when the number of investable assets is two or three. Since personal tax is similar to an extremely high transaction cost, we surmised the conclusion of transaction cost research, which is applicable to the case of personal tax. This research concluded that the whole state space can be split into two regions, a no-trading region and a trading region, see Shreve and Soner (1994). For the no-trading region, the transaction cost is too large to be counteracted by the benefit of moving the current portfolio toward the optimal portfolio. For the trading region, the investor should trade enough to move the portfolio to the boundary, which separates the no-trading region and the trading region. The boundary can be described by super contact conditions, which are first-order partial differential equations. When the investable assets are stock and cash only, the no-trading region can be described solely by the weight of stock. It is an interval around the optimal stock weight corresponding to the no-transaction cost case. The trading instruction is clear in this instance: buy or sell enough stock to reach the ends of the interval.

In the case of searching for the analytic solution, the common approach is to guess the form of the value function. The coefficients in the value function are fitted with the optimality conditions for the no-trading zone (Hamilton–Jacob–Bellman equation) and super contact conditions at the boundary. When there are more than two investable assets, it is almost impossible to guess the form. The shape of the boundary separating the no-trading and the trading zones is much harder to define by functions. Researchers usually assume the non-trading zone to be a cubic. Leland (2000) carried out such an approach in the case where the utility function itself is quadratic. The guessed value function is comprised of numerous summations of the products of different power functions. There could be as many as 2^N combinations of trading directions, hence 2^N super contact conditions, where N is the number of investable assets. This makes it impossible to get the right value function. The cubic no-trading zone assumption cannot be verified to be true in high dimensional cases. Even if it is the right description of the no-trading zone in a purely constant transaction cost case, I doubt it will be true in our case because the transaction cost (tax) varies depending on the tax base of each asset. Notice, even though the Leland's utility function is quadratic, the value function is much more complicated. Markowitz and van Dijk (2003) argue that this complicated value function can be approximated by a quadratic function. Instead of searching for the "true" optimal solution, investors should look for the near optimal solution based on the approximated quadratic value functions. Markowitz and van Dijk show that the near optimal solution is as good as the true optimal solution in a case that has a total of 55 states. Markowitz and van Dijk's method is fast and scalable. Kritzman *et al.* (2009) apply this methodology to the portfolio rebalancing problem when the number of assets is in the thousands. It shows that Markowitz and van Dijk's heuristic method produces a better solution than the standard dynamic programming with a coarse grid when the number of assets is moderate. When the number of assets is large, there is no dynamic solution.

This paper is in the spirit of Markowitz and van Dijk. Instead of assuming a quadratic utility function, we assume the traditional power utility function and approximate the value function via quadratic function. We will introduce two state variables for each investable instrument, namely the cost and its capital gain. Therefore, there will be $2 * N$ state variables, while N is the number of investable instruments. The approximated optimal value function is a simple quadratic function of these $2 * N$ variables. Instead of fitting the quadratic function's coefficients by super contact conditions and the HJB equation, we focus on the utility values of these $2N + 1$ specified states. The first N utility values are derived from fully taxed dollars for each asset. The second N utility values are derived from unit dollar capital gains for each asset. The last one is the utility value of the optimal portfolio. We then construct a quadratic utility function that coincides with these utility values. This approximation takes into account the optimal portfolio's weights, the optimal portfolio's utility value, the capital gains of each asset, and the utility value of individual assets. With this quadratic approximation, we can derive the portfolio rebalancing rule, which is applicable to any portfolio with any capital gain distribution. The rebalancing rule can be derived easily as the best trade-off between paying tax now vs. the cumulative benefit captured by the approximated quadratic value function.

We focus on the consumption problem for a retiree's decumulation phase. The technique described here is applicable to the lifecycle's cumulative phase as well. The retiree pays tax when withdrawing assets for consumption, as well as when rebalancing the current portfolio to an optimal portfolio. To solve the timing issue of capital gain realization, we assume the investor withdraws assets following a predefined time-dependent fraction of his total assets in the consumption period. The withdrawn amount is spread proportionally to all assets. The investor's utility is measured by consuming the after-tax amount of his withdrawal. This withdrawal amount is comparable to an annuity payout calculation with fixed maturity. It is the optimal withdrawal amount when there is no tax and other market restrictions.

If we know the realization of asset return and the withdrawal amount, the tax calculation is straightforward. This makes Monte-Carlo simulation a powerful tool in calculating the utility of multi-period consumptions for any un-rebalanced portfolio. This is how the above-mentioned $2 * N + 1$ utilities are calculated.

2 Market dynamic

We assume there are N investable assets, whose returns at period t is

$$R_{i,t} = \mu_{i,t} + \varepsilon_{i,t} \tag{1}$$

where μ_t is the vector of expected returns, and ε_t is a vector of random returns with mean zero and variance–covariance matrix Σ_t. The realized returns consist of dividends, short term, and long-term capital gains distribution. We apply the appropriate tax rate for each component of return. An independent identical distribution assumption for the asset returns is not needed in this paper. We assume that the expected utility from consumptions is

$$U(t, \tilde{c}) = E \sum_{s=t}^{T} D^{s-t} U(c_s) \tag{2}$$

where T is the fixed planning period and c_s is the consumption for period s.

We constrain the consumption/portfolio process $(c_{\mathrm{t}}, \pi_{\mathrm{t}})$, $0 \leq t \leq T$, such that

(1) The consumption process c_{t} must be non-negative,

$$0 \leq c_t, \quad 0 \leq t \leq T \tag{3}$$

(2) The retiree cannot short sell any security, i.e., the portion allocated to each asset must satisfy

$$\pi_t^i \geq 0, \quad 0 \leq t \leq T \tag{4}$$

(3) Wealth remains non-negative, that is,

$$X_t \geq 0, \quad 0 \leq t \leq T \tag{5}$$

The change of wealth for the period is given by

$$\Delta X_t = (\mu_t' \pi_t - c_t + \pi_t' \varepsilon_t - tax_t), \quad 0 \leq t \leq T \tag{6}$$

In real-time investment decisions, there are constraints added for portfolio π. Here, we assume budget constraint, e.g., you can't borrow money to invest. In this case,

$$\sum_{i=1}^{N} \pi_t^i = x_t \tag{7}$$

Let us say you are a U.S. investor who would like to hold foreign stocks to diversify, but you don't want your retirement income to depend largely on the outcome of foreign equity markets. In this paper, we constrain our foreign equity exposure to no more than 25 percent. The budget constraint and additional linear constraints are the standard constraints for a one-period mean–variance portfolio optimization problem. The value function is the maximum achievable utility, i.e.,

$$\begin{aligned} V(t, \pi_t) &= \max_{(\pi_t, c_t)} U(t, \tilde{c}) \\ &= \max_{(\pi_t, c_t)} E \sum_{s=t}^{T} D^{s-t} * U(c_s) \end{aligned} \tag{8}$$

If the market is complete, the total wealth x is a sufficient state since you can redistribute your wealth without cost. Therefore, the value function becomes

$$V(t, \pi_t) = V(t, x) \tag{9}$$

$V(t, x)$ can be solved backwardly since

$$V(t, x) = \max_{\pi_t, c_t} \{U(c_t) + D * V(t+1, x_{t+1})\}$$

where x_{t+1} is the ending wealth by choosing the portfolio/consumption combination (π_t, c_t) as defined by Equation (6). Since the value function is two dimensional, it is relatively easy to solve numerically in the complete market case.

In this paper, we assume that the utility function used in Equation (2) is a power function

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{10}$$

where γ is the risk-aversion parameter. With this power utility function, the value function will have the homogeneity property, i.e.,

$$V(t, \lambda * \pi_t) = \lambda^{1-\gamma} * V(t, \pi_t) \qquad (11)$$

With the homogeneity property, we need only to calculate the optimal utility value of states π_t whose weights added up to one, which is the corresponding budget constraint of our portfolio rebalancing problem. The homogeneity property makes the rebalancing problem easier.

3 Withdrawal assumption

Within the context of a complete market, the optimal consumption amount c_t at time t is comparable to the annuity payout by spending the total wealth x_t at time t to purchase a fixed annuity with maturity at T. For each dollar premium, the annuity payout calculation takes this form,

$$K_t = \frac{1}{1 - e^{-g(T-t)}} \qquad (12)$$

where g is the assumed rate of growth. Xu and Shreve (1992) showed that the appropriate choice of g for utility maximization is

$$g = \frac{\beta}{\gamma} + r\left(1 - \frac{1}{\gamma}\right) + 0.5 * \theta^2 * \frac{1}{\gamma} * \left(1 - \frac{1}{\gamma}\right) \qquad (13)$$

where β is the utility discount factor, which is equal to our $-\ln(D)$ of Equation (2), and θ is the price of risk, which is the excess return divided by the risk. Laibson *et al.* (2004) estimated the long-term discount factor D to be 0.962. In this paper, we will choose the risk aversion parameter γ to be 3, signifying a conservative level of risk. If γ were 2, then one-period stock-cash mean–variance analysis (using data in Exhibit 2) would allocate 85 percent in stock and 15 percent in cash, which it is too risky for retirees. With γ being 3, the one-period mean–variance investor will allocate 57 percent to stock and 43 percent to cash, which is consistent with the allocations of 65 years old retiree reported by Xu (2015). If we assume the real short-term rate to be zero as it is now, then the calculated g of Equation (13) is 0.027. Xu and Anichini (2016) recommend g to be 0.01 by taking into consideration other practical factors, e.g., the uncertainty of the price of risk and the probability of spending much less in later retirement years. This is the assumption used in this paper. Blay and Markowitz (2016) use the assumption of $g = 0$ to calculate the efficient frontier of present values of future consumption.

4 Quadratic approximation of value function for arbitrary portfolio

Let us define the utility value of one dollar of cost and one dollar of capital gain for asset i at time t as $U^{i,c}(t)$ and $U^{i,g}(t)$. These are the utility values of un-rebalanced portfolios consisting of one dollar of cost (respectively capital gain) of asset i and zero dollars of other assets. In this paper, we assume that you sell the capital gain and cost proportionally when you sell the asset. Let $MV(t)$ be the maximum utility achieved at time t, which is approximated as the utility of an un-rebalanced optimal portfolio.

Let π_t^i be the portfolio weight of asset i. For each asset, we introduce two state variables, its capital gain $\pi_t^{i,g}$ (capital loss, in case $\pi_t^{i,g}$ being negative) and its cost $\pi_t^{i,c}$. By definition,

$$\pi_t^i = \pi_t^{i,c} + \pi_t^{i,g}$$

We conjecture that the value function for any portfolio can be approximated by

$$V(t, \pi_t) = MV(t) - TD_t - PD_t \qquad (14)$$

where $-TD_t$ is the utility deduction because of the portfolio's tax liability, and PD_t is the utility deduction from holding a non-optimal portfolio. We can visualize the approximated value function as an umbrella with the optimal portfolio at the highest point, and individual assets' utilities as the edges. By the convexity of the value function, we can argue that the first partial derivative of the value function should be zero at the optimal portfolio point. This implies PD_t and TD_t should be quadratic only. The simplest quadratic function of TD_t will be

$$TD_t = \sum_{i=1}^{N} \pi^{i,g}(t) * \pi^{i,g}(t) * (U^{i,c}(t) - U^{i,g}(t)) \tag{15}$$

The simplest quadratic function of PD_t will be

$$PD_t = \sum_{i=1}^{N} (\pi^i(t) - \pi^{i,o}(t)) * (\pi^i(t) - \pi^{i,o}(t)) * (MV(t) - U^{i,c}(t)) \tag{16}$$

where $\pi^{i,o}(t)$ is the weight of asset i of the optimal portfolio.

The homogeneity property (Equation (11)) implies that

$$V(t, \pi_t) \approx \max_{\{\lambda\}} \left\{ 1 - tax\_rate * \sum_{i=1}^{N} \lambda * \pi^{i,g}(t) \right\}^{1-\gamma} * V(t, \pi_t(1-\lambda) + \lambda * \pi^o(t)) \tag{17}$$

where λ is the parameter of selling a portion of the current portfolio. The first term is the tax cost effect of trading. The second term is the optimal utility value of a normalized portfolio after the trade. To shorten the notation, let us define the potential tax liability of selling the whole portfolio as

$$PTax_t = tax\_rate * \sum_{i=1}^{N} \pi^{i,g}(t) \tag{18}$$

To find the trade-off analytically, we replaced the second term of Equation (17) through quadratic approximation (Equation (14)). By applying linear and quadratic approximation to Equation (17), we have

$$V(t, \pi_t) \approx \max_{\{\lambda\}} \left\{ \begin{array}{l} MV(t) * [1 - (1-\gamma) * \lambda \\ * PTax_t - 0.5 * (1-\gamma) * \gamma \\ * (\lambda * PTax_t)^2] + (1-\lambda)^2 \\ * [TD_t + PD_t] \end{array} \right\} \tag{19}$$

The first term is the approximated tax effect of rebalancing the portfolio, and the second term is the benefit of rebalancing the portfolio. The tax effect depends on the tax rate, the capital gains in the portfolio, and the risk aversion parameter, and includes both the linear and quadratic terms commensurate with the amount selling. The benefit consists of the quadratic term only, commensurate with the amount selling. The optimal portion $\bar{\lambda}$ to sell is obtained by

$$1 - \bar{\lambda} = \frac{(1-\gamma) * MV(t) * (PTax_t + \gamma * (PTax_t)^2)}{2 * (TD_t + PD_t) + (1-\gamma) * \gamma * MV(t) * ((PTax_t)^2)} \tag{20}$$

where $1 - \bar{\lambda}$ is the optimal portion to keep in the current portfolio. If $\bar{\lambda}$ is less than or equal to zero by Equation (20), then the tax cost is higher than the benefit. The best action, in this case, is no trading. By rewriting the non-negativity constraint of Equation (20), the no-trading zone can be characterized by parabolic equation:

$$2 * (TD_t + PD_t) \leq (1-\gamma) * MV(t) * PTax_t \tag{21}$$

The left side of the Equation (21) is the benefit of trading, and the right side is the adjusted maximum cost of trading.

5 Empirical results

Since there is no analytical solution for this problem, we will resort to Monte-Carlo simulations. Let us assume a retiree's holdings are in mutual funds and individual stocks. The mutual funds

Exhibit 1: Asset classes.

Asset class	Asset class name	Notation	Proxy name
1	Cash	Csh	Citi Treasury Bill 3 Month Index
2	U.S. Intermediate Term Bond	IB	Barclays Govt/Credit Intermediate Term Bond Index
3	U.S. Long Term Bond	LB	Barclays Govt/Credit Long Term Bond Index
4	U.S. High Yield Bond	HY	Credit Suisse High Yield Index
5	U.S. Large Cap Stock	LS	S&P 500 index
6	U.S. Mid Cap Stock	MS	S&P 400 index
7	U.S. Small Cap Stock	SS	Russell 2000 index
8	International Stock	EAF	MSCI EAFE Index
9	Emerging Market	EM	MSCI EM Index
10	Individual Stock	IS	

Exhibit 2: The mean and standard deviation of sample returns at starting period.

	Csh	IB	LB	HY	LS	MS	SS	EAF	EM	IS
Mean	3.51	4.59	5.28	6.95	9.04	9.76	10.59	9.02	12.89	9.14
Std. dev.	1.30	3.95	10.20	10.63	18.60	21.89	24.41	21.89	30.22	39.26

are classified into nine asset classes as shown in Exhibit 1. Exhibit 2 displays the sample mean and sample standard deviations at the beginning of the 30 years' returns for each asset class. We use the CIR model to generate cash and bond returns, which introduce some auto-correlations. We use Pearson type IV distribution to model the equity returns, which are more consistent with observed returns, see Markowitz and Usemen (1996).

Even though the utility maximization is the *de facto* theorectical model, researchers do not report utility values directly because they are subject to rescaling. In this paper, we will report relative utility ratios and certainty equivalent return (CER). For each utility value, we define the CER to be the fixed return, which will generate the same utility subjected to the same withdrawal and tax rule. In the complete market case (no tax and no constraints), we know that the portfolio maximizing the expected multi-period utility is the same portfolio which maximizes the one-period risk-adjusted return with the same risk aversion parameter. It is true even in the presence of a transaction (trading) cost. In the case of constrained portfolio optimization, we do not know whether that is still true, but we assume that the optimal portfolio is on the "one-period" mean–variance efficient frontier. We find the optimal portfolio by searching all the portfolios on this efficient frontier. Exhibit 3 shows the optimal portfolio weights for various planning periods.

Please note that the optimal portfolios have 50 to 60 percent weight in bonds, which is in the

Exhibit 3: Utility maximization portfolio weights with $\gamma = 3$.

Yrs	Csh	IB	LB	HY	LS	MS	SS	EAF	EM
10	7	53	0	0	0	0	17	0	23
20	0	62	0	0	0	15	6	0	17
30	0	59	0	0	0	16	0	0	25

holding range of bonds for a 65-year old retiree, see Xu (2015).

First, we report the CERs for individual asset classes and three special portfolios.

The certainty equivalent return severely penalizes the volatility.The emerging market (EM) and individual stock (IS) asset classes,which have very high volatility, have negative CERs. The bonds have higher CERs than the equities. Not surprisingly, the optimal portfolio (Opt) has the highest certainty equivalent return. The equal-weighted portfolio (AW) of all assets (from asset 1 to asset 9) has the second highest certainty equivalent return. The equity (from asset 5 to asset 9) equal-weighted (EW) portfolio has lower CER than bonds.The CER of Opt is 25 basis points more than the AW's CER. Exhibit 4a clearly illustrates the benefits of diversification.

Next, we will report the CERs for assets with 50 percent capital gains. For the AW portfolio, the actual capital gain is 27 percent for all those exhibits titled with 50 percent capital gains because we assume bonds do not have capital gains. The AW's actual capital gain is 57 percent for exhibits titled with 100 percent capital gains for the same reason. That is why we put a star on AW notation.

In this simulation, the long-term tax rate is assumed to be 20 percent. The 50 percent capital gains implies a 10 percent deduction of return over the planning period. By comparing the CERs of Exhibits 4a and 4b, we see that the tax effect is larger for shorter periods. It is one hundred basis points for a 10-year planning period, and 40 basis points for a 30-year planning period.

Another way to make sense of the utility values is to calculate the utility ratios. We define the portfolio diversification benefit (DB) relative to an undiversified portfolio as the ratio of the utility of the undiversified portfolios to the utility of the optimal portfolio

$$DB_t^i = \frac{U^{i,c}(t)}{MV(t)} - 1$$

This definition takes into account the negativeness of $MV(t)$ and $U^{i,c}(t)$ because $\gamma = 3$ and $abs(U^{i,c}(t)) > abs(MV(t))$. We report the relative benefit of diversifaction in Exhibit 5.

Exhibit 5 confirms our intuition with respect to diversification. The individual stock (IS) is the least diversified asset in our study and the utility

Exhibit 4a: Certainty equivalent returns for no capital gain assets at starting point.

Yrs	Opt	Csh	IB	LB	HY	LS	MS	SS	EAF	EM	IS	AW	EW
10	5.19	3.82	4.17	3.11	3.53	3.44	2.26	1.21	0.99	−1.56	−9.97	4.81	2.78
20	5.35	3.80	4.23	3.34	3.69	3.56	2.72	1.60	1.43	−0.90	−8.91	4.98	3.17
30	5.25	3.64	4.11	3.20	3.63	3.58	2.57	1.63	1.13	−0.77	−8.90	4.92	3.17

Exhibit 4b: CERs with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	1.60	0.77	0.14	−0.06	−2.17	−10.19	3.78	1.18
20	2.74	2.02	1.11	0.99	−1.05	−8.93	4.47	2.46
30	3.05	2.17	1.37	0.95	−0.81	−8.90	4.55	2.73

Exhibit 5: The utility benefit of optimal portfolio to undiversified portfolios in percentage.

Yrs	Csh	IB	LB	HY	LS	MS	SS	EAF	EM	IS	AW*	EW
10	10	7	16	12	13	23	33	36	77	416	3	19
20	21	14	28	22	24	39	62	66	161	2219	4	31
30	29	19	39	29	30	56	87	108	263	13007	5	40

benefit of the optimal portfolio is 13007 percent over it. The AW is the most diversified asset in our study, so the utility benefit of the optimal portfolio over it is less than 5 percent.

In order to relate the utility improvement to the certainty equivalent return, we perform the following regression analysis using data from Exhibits 4a and 5

$$Opt_t - CER_t^i = \beta * \text{In}(DB_t^i + 1) + \varepsilon$$

The analysis shows that β is 0.05 with a t-statistic of 40 for 30-year planning data. The β is 0.07 with t-statistics 190 for 20-year planning data. Each one percent improvement in utility corresponds to 5–7 basis points more certainty equivalent return.

There are two simple ways of handling tax. One way is to hold an asset as long as possible (and sell it for consumption purposes only). The other way is to immediately realize all the capital gains, pay the tax, and reinvest in the same security. We can define the tax delay benefit by calculating the utility ratio of these two strategies:

$$TDB_t^i = \frac{U^{i,c}(t) * (1 - tax)^{1-\gamma}}{U^{i,g}(t)} - 1$$

This definition takes into account the negative values of $U^{i,c}(t)$ and $U^{i,g}(t)$. We report the benefits of delaying tax in Exhibit 6.

Exhibit 6 shows that all portfolios benefit from not realizing capital gains immediately. Exhibit 5 demonstrates that doing nothing is not the best policy because it ignores the diversification benefits. The prudent tax strategy is to manage the tax by trading some amount of the holding. We report the optimal keep amount in Exhibits 7a and 7b by numerically searching λ directly using Equation (17). One minus the optimal keep amount is the optimal trading amount.

Exhibit 6: Utility benefit of delaying tax in percentage with 100 capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	7	7	7	8	7	1	6	8
20	11	10	11	12	15	0	8	12
30	14	13	15	19	30	0	9	16

The tax costs reported in parenthesis are the initial tax cost and sum of subsequent tax costs for adjusting the portfolio. The subsequent tax costs are quite large because the X percent trading method requires trading every year. Exhibits 7a and 7b substantiate that the optimal keep amount does not vary much with the planning period or amount of capital gains. Exhibits 7a and 7b contradict our intuition that you would trade more if the planning period is longer because it has more time to catch the diversification benefits. If we think of the tax as a trading cost, the optimal keep amount in Exhibit 7b is slightly lower than the optimal keep amount in Exhibit 7a. That is, we trade more if there is less tax to pay. Please note that for the all asset equal-weighted portfolio (AW*), the best rebalancing rule is to trade 10 percent per year.

Now we report the derived rebalancing rule from the quadratic approximation of the utility function.

Exhibit 7a: Optimal keep amount and rebalancing tax costs (in parenthesis) in percentage with 100 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	70	60	60	50	40	30	90	70
	(6,11)	(8,11)	(8,11)	(10,9)	(12,8)	(14,6)	(1,5)	(6,12)
20	70	60	60	50	50	30	90	70
	(6,16)	(8,14)	(8,14)	(10,12)	(10,12)	(14,8)	(1,10)	(6,16)
30	70	70	60	50	50	30	90	70
	(6,17)	(6,17)	(8,15)	(10,12)	(10,12)	(14,10)	(1,14)	(6,17)

Exhibit 7b: Optimal keep amount and rebalancing tax costs (in parenthesis) in percentage with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	60	50	50	40	30	20	90	60
	(4,6)	(5,6)	(5,6)	(6,5)	(7,4)	(8,3)	(1,3)	(4,7)
20	70	60	50	40	30	20	90	60
	(3,10)	(4,9)	(5,8)	(6,7)	(7,6)	(8,5)	(1,7)	(4,10)
30	70	60	60	40	40	20	90	70
	(3,13)	(4,13)	(4,13)	(6,10)	(6,10)	(8,7)	(1,11)	(3,14)

The QP optimal initial keep amount (Exhibits 7c and 7d) is very different from optimal keep amount reported in Exhibits 7a and 7b for undiversified portfolios represented as single asset class. The difference is smaller for the diversified portfolios (AW* and EW). The QP heuristic method incurs a larger initial tax cost compared to the *X* percent heuristic method. But there is almost no tax cost of maintaining the portfolio at the boundary of no trading zone. The total tax cost is smaller with QP heuristic method. We report the utilities' ratios achieved by QP over the best of *X* percent method. If the ratio is 100 percent, then QP achieves the same utility. If the ratio is

Exhibit 7c: QP optimal initial keep amount and rebalancing tax costs (in parenthesis) in percentage with 100 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	38	32	31	27	23	4	100	100
	(12,0)	(14,0)	(14,0)	(15,0)	(15,0)	(19,0)	(0,0)	(0,0)
20	33	29	22	21	14	1	100	89
	(13,0)	(14,0)	(16,0)	(16,0)	(17,0)	(20,0)	(0,0)	(2,0)
30	28	24	17	15	13	0	70	78
	(14,0)	(15,0)	(16,0)	(17,0)	(17,0)	(20,0)	(3,0)	(4,0)

Exhibit 7d: QP optimal initial keep amount and rebalancing tax costs (in parenthesis) in percentage with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	36	27	27	21	18	3	100	100
	(6,0)	(7,0)	(7,0)	(8,0)	(8,0)	(10,0)	(0,0)	(0,0)
20	25	23	15	13	9	1	77	76
	(8,0)	(8,0)	(9,0)	(9,0)	(9,0)	(10,0)	(1,0)	(2,0)
30	19	16	10	9	7	0	42	57
	(8,0)	(8,0)	(9,0)	(9,0)	(9,0)	(10,0)	(3,0)	(4,0)

greater than 100 percent, then QP performs better. If the ratio is less than 100 percent, then QP underperforms.

Exhibit 8a substantiates that, in general, a QP derived rebalancing rule performs as well as the best *X* percent heuristic rebalancing rule. The notable exception is the case for an all equity equal-weighted (EW) portfolio. It underperforms considerably for the short-term planning period. The underlying reason is that the QP derived rule recommends no-rebalancing when the planning period is short, see Exhibit 7c. Next, we will report the efficiency of QP with different capital gains (Exhibit 8b).

In this more realistic capital gain case, the QP performs better in most cases. Even for the EW portfolio, the underperformance has been reduced by half for short planning periods. Since CER is more intuitive, we will report the CER for QR rebalanced optimal portfolios (Exhibit 9).

Exhibit 8a: Efficiency of QP over *X* percent trading with 100 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	100	100	100	100	100	100	99	86
20	100	101	101	101	101	100	99	90
30	100	100	100	100	99	100	100	96

Exhibit 8b: Efficiency of QP over *X* percent trading with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	100	100	100	100	101	101	100	92
20	101	101	101	101	101	101	100	97
30	100	101	101	101	101	101	100	100

Exhibit 9: CER of QR rebalanced optimal portfolio with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	2.53	2.46	2.48	2.34	2.37	2.32	3.86	2.48
20	3.84	3.84	3.81	3.74	3.77	3.72	4.61	3.76
30	4.07	4.06	4.04	4.00	4.00	3.96	4.68	4.04

Please note that the CERs are not much different across the rebalanced portfolios. AW's CERs are much higher because its actual capital gains is 27%. These CERs are much higher than the CERs of un-rebalanced portfolios as reported in Exhibit 4b. In order to calculate the benefit of prudent tax management, we first calculate the CERs for tax ignorant rebalancing rule (rebalancing to the optimal portfolio completely at beginning of each period), and substract it from the CERs of Exhibit 9.

By managing tax prudently, we can save the retiree almost 15–20 basis points of return (Exhibit 10).

Exhibit 10: CERs of tax prudent over tax ignorant rebalancing with 50 percent capital gains.

Yrs	LS	MS	SS	EAF	EM	IS	AW*	EW
10	0.30	0.23	0.25	0.11	0.14	0.09	0.31	0.25
20	0.26	0.26	0.23	0.16	0.19	0.14	0.29	0.18
30	0.17	0.16	0.14	0.10	0.10	0.06	0.19	0.14

6 Conclusion

We study two important issues related to portfolio management in the presence of personal tax. The first issue is the urgency of adjusting a portfolio with capital gains toward a more suitable portfolio. The answer depends on the capital gains of the portfolio, the risk aversion, and the difference in the current portfolio compared to the optimal portfolio. If you adopt trading X percent rule, adjustments should be made over a long period of time (10% turnover per year) if the portfolio is close to the optimal portfolio like all asset class equal-weighted portfolio. It should be done urgently (80% turnover per year) if the asset is risky individual company stocks. The second issue is whether there are better rebalancing rules. The answer is yes. We can approximate the underlying value function by a quadratic function and derive a rebalancing rule based on this approximation. This derived rebalancing rule recommends a similar initial trading amount for diversified portfolios and company stock in a realistic case of 50% capital gains. It achieves more utility than the best trading X percent anually rule in most cases. Finally we show that the it adds about 20 basis points of certainty equivalent returns when compared to the tax ignorant rebalancing rule in most cases.

Acknowledgement

The author would like to thank Harry Markowitz and an anonymous referee for insightful comments and Deirdre Myers' helping in this manuscript.

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Keywords: Tax; capital gains; portfolio rebalancing; multi-period; HJB equations; no-trading zone