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EXPONENTIAL GLIDE PATHS*

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In the absence of market-timing ability, investors are better-off keeping their asset allocation constant through time. Target-date funds help reduce variation in the asset allocation, by taking into account that human capital, which is a part of the investor's total portfolio and is typically considered to be bond-like, diminishes with age. To compensate, target-date funds reduce the allocation to equities in the financial portfolio over time. Funds almost universally do so in a linear fashion, following straight-line glide paths. We show that linear glide paths imply two systematic deviations from constant asset allocation, and suggest a simple correction, the exponential glide path. Exponential glide paths lead to a typical increase of 5–22% in welfare relative to linear glide paths.

1 Introduction

Unless they possess an ability to time the market, investors are better-off keeping their asset allocation constant over time, rather than varying it from one period to another. This statement was originally proven for Constant Relative Risk Aversion (CRRA) preferences (Leland, 1969; Merton, 1969; Samuelson, 1969, 1989a, 1990, 1997), and was recently extended to *all* risk averters (Levy and Levy, 2021). This result has important implications for lifecycle investing. The investors' total portfolio includes not only financial assets, but also human capital, which can be thought of as the present value of all future income from labor (the total portfolio may include other elements as well, such as real estate). Human capital is typically more bond-like than stock-like. As the value of this human capital component decreases with age, if a constant proportion of the *total* wealth is allocated to stocks, this implies that the proportion of stocks in the *observable* (i.e. financial) portfolio should decrease with age.

A very popular way to reduce the allocation to equities over time is through investment in target-date funds (also called lifecycle funds), which employ an "automatic pilot" equity reduction glide path. Indeed, the lifecycle mutual fund industry has grown dramatically in recent decades, managing approximately 1.9 trillion

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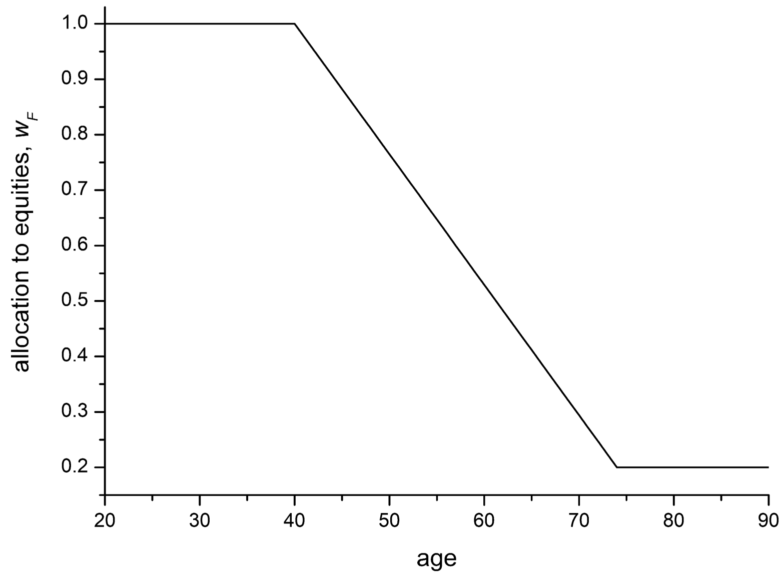

Figure 1 A typical linear glide path employed by most target-date funds (see, for example, Ibbotson, 2008, Figure 5). The allocation to equities is shown as a function of age. Most funds start with an allocation of 100% to equities for several years, and then reduce the allocation linearly until the final allocation level is reached. The figure shows the case where at ages 20–40 the allocation to equities is 100%, between the ages of 40 and 75 the allocation decreases linearly from 100% to 20%, and it remains at 20% for ages 75–90.

dollars.[1] Almost all target-date funds reduce the allocation to equities linearly over time (see, for example, Ibbotson, 2008). Figure 1 depicts a typical glide path employed. The linear glide path is simple, and resonates with popular rules-of-thumb such as the rule stating that the percentage allocated to stocks should be 100 minus age. Although glide paths differ from one target date fund to another, the principle is the same: the longer the investment horizon, the larger the investment weight in risky assets. The decline in the weight to risky assets is almost universally linear.

While target-date funds are helpful in reducing the variation in the asset allocation in the total portfolio, in a recent paper (Levy and Levy, 2021) we have shown that they suffer from two systematic deviations relative to the constant-allocation goal. The purpose of the present paper is to focus on the practical implications of the Levy and Levy (2021) paper, and in particular, to provide a simple and realistic alternative to the linear glide path, which may substantially improve investors' welfare.

The first systematic deviation from the optimal constant allocation induced by the current practice is that most funds do not adjust the allocation to stocks in response to market fluctuations. However, they should, if they are to maintain the optimal policy of a constant allocation in the total portfolio. The reason is that when the market fluctuates, the size of the financial portfolio fluctuates, and so does its weight relative to the human capital component in the total portfolio. In order to maintain a constant allocation in the total portfolio, the allocation in the financial portfolio (i.e. the target-date fund) must be adjusted in response to market fluctuations. Theoretically, these adjustments should be made daily, or at even higher frequency. Form a practical perspective, however, this is not necessary—we show below that annual adjustments suffice to capture most of the adjustment benefits.

The second systematic deviation from constant allocation is due to the *linearity* of the employed glide paths. We show that the average allocation to equities should decrease as an *exponential* function of time. The intuition for this result is that while the human capital component in the total portfolio decreases over time, the equity component increases (on average) exponentially, due to compounding effects. We provide an exact formula for the optimal exponentially decreasing glide path that ensures (on average) a constant allocation in the total portfolio.

Thus, both of the systematic biases that target-date funds suffer are easy to correct. The welfare improvements obtained by implementing these two corrections vary depending on the investor's degree of risk aversion and on the initial ratio of human capital to total wealth. For the typical empirical values of income, savings and wealth by age, and risk aversion, we estimate that the increase in welfare obtained by correcting both biases is in the range of 15–30%, depending on the degree of risk aversion and on the initial human capital component. By only switching from a linear glide path to the appropriate exponential glide path (i.e. without adjustments to market fluctuations) welfare increases by 5–22%. These improvements are economically quite substantial. Thus, establishing target-date funds with the exponentially decreasing glide path could potentially dramatically improve investors' retirement conditions.

2 The Systematic Biases Implied by Linear Glide Paths

While the logic of target-date funds is sound, and they do help reduce the variation in the asset allocation in the total portfolio, they imply two systematic biases relative to the optimal constant-allocation policy (in the total portfolio):

(B1) No adjustments to market fluctuations. The predetermined glide path does not adjust for market fluctuations. If the market goes down, the weight of the financial capital in the total portfolio decreases, and therefore, in order to keep the allocation constant, one should increase the allocation to equities in the financial portfolio (and the exact opposite when the market goes up). For example, suppose that an investor has \$50 of financial wealth invested in a target-date fund, and \$50 of human capital (HC). The fund is invested 80% in equities. For simplicity, assume that HC is bond-like. Thus, the proportions in the investor's total portfolio are 40% in stocks and 60% in bonds (including the HC component). To be specific, the investor holds \$40 in stocks and \$60 in bonds (\$10 in the fund + \$50 HC). Now suppose that in a given year the stock market goes down by 20%, and the bond value does not change. The investor's stocks are now worth only \$32, his financial portfolio is worth \$42, and his total portfolio value is \$92 (\$32 stocks + \$10 bonds + \$50 HC). If the fund does not adjust its allocation in response to this price fluctuation, and rebalances to keep the same proportion of 80% in equity, the investor will now hold $0.80 \times 42 = \$33.6$ in stocks, and \$60 in bonds, i.e. the allocation to stocks in the total portfolio is reduced to 36.5% ($33.6/92 = 0.365$). Thus, the equity allocation in the total portfolio has been reduced from 40% to 36.5%, because the relative weight of the financial portfolio has shrunk relative to the weight of HC. In order to keep a constant allocation in the total portfolio, the fund must increase its allocation to equity (in this case to 87.6%, so that the equity proportion in the total portfolio remains $40\% = 0.876 \times 42/92$).

(B2) Linearity. As we show below, the average weight of human capital in the total portfolio decreases as an *exponential* function of time, implying that the optimal reduction in the allocation to equities in the financial portfolio should also be exponential. Otherwise, the optimal constant allocation is not achieved.

Let us elaborate. Most studies on lifecycle investing investigate the joint problem of optimal investing and optimal consumption. The solution to this problem requires dynamic programming, and therefore exact knowledge about the investor's utility function. Our setting is somewhat different: on the one hand the results are general, and are preference-free; on the other hand, we do not solve for the optimal *level* of the average asset allocation or for the optimal consumption path. In our framework, we take the average asset allocation and the consumption path as given, and identical across strategies (linear or exponential glide path), and we compare the certainty equivalent of terminal wealth for the different strategies. Obviously, if under these conditions one strategy yields a higher certainty equivalent of terminal wealth, this advantage can be converted to the same certainty equivalent of terminal wealth, but with higher consumption along the path. As the consumption path is taken as given, we ignore the utility derived from intermediate consumption, which is the same across strategies. For our purposes, it is convenient to define human capital as the present value of all future labor income minus consumption, i.e. the present value of all future savings.[2]

We denote the investor's financial wealth by F, his human capital wealth by HC, and his total wealth by $TW = F + HC$. Assuming that human capital is bond-like[3] (or alternatively, that HC represents the bond-like component of human capital), the allocation to equities in the total portfolio is given by:

$$w_{TOT} = \frac{w_F \cdot F}{TW} = \frac{w_F \cdot F}{F + HC}, \qquad (1)$$

where w_{TOT} denotes the allocation to equities in the *total* portfolio, and w_F denotes the allocation to equities in the *financial* portfolio. Alternatively, we can express w_F as a function of w_{TOT}:

$$w_F = \frac{w_{TOT} \cdot TW}{F} = w_{TOT}\left(1 + \frac{HC}{F}\right). \qquad (2)$$

In order to keep w_{TOT} at a constant level c, w_F should change with F and HC according to:

$$w_F = c \cdot \frac{TW}{F} = c \cdot \left(1 + \frac{HC}{F}\right). \qquad (3)$$

Equation (3) mathematically expresses the two systematic biases that are induced by predetermined linear glide paths:

(B1) Financial wealth F fluctuates with the market; therefore, so should the allocation to equities in the financial portfolio w_F—when the market goes up, w_F should be reduced, and vice versa.

(B2) Due to compounding effects, the average financial wealth F increases exponentially, while HC decreases over time. The ratio $\frac{HC}{F}$, *on average* decreases as an exponential, rather than a linear, function of time; therefore, so should w_F. In the next section we provide the exact formula for the exponential glide path that ensures a constant asset allocation (on average).

3 The Exponential Glide Path

Levy and Levy (2021) show that in order to keep the average asset allocation in the total portfolio (including human capital) constant over time at a value c, the allocation to stocks in the financial portfolio should decrease as an exponential function of time, given by:

$$w_F(t) = \frac{c}{1 - e^{-c(r_m - r_f)t}\left(\frac{HC_0}{TW_0}\right)\left(\frac{e^{r_f(T-t)}-1}{e^{r_f T}-1}\right)}, \tag{4}$$

where r_f is the continuous risk-free return, r_m is the continuous expected return on equities, $\left(\frac{HC_0}{TW_0}\right)$ is the initial proportion of human capital in the total portfolio, and T is the terminal date (the derivation of Equation (4) is provided in Levy and Levy, 2021). The following points about the solution in Equation (4) are worthwhile noting:

(i) Equation (4) is the solution to the continuous-time problem, where the asset allocation is adjusted continuously. However, it serves as an excellent approximation for the discrete-time solution, as discussed below.

(ii) As $t \to T$ the right brackets in the denominator approach 0, and therefore the denominator approaches 1, and we have $w_F(t) \underset{t \to T}{\longrightarrow} c$. This is as expected, because as $t \to T$ the human capital shrinks to 0, and the financial portfolio coincides with the total portfolio.

(iii) At $t = 0$ we have: $w_F(0) = \frac{c}{1-\left(\frac{HC_0}{TW_0}\right)} = \frac{c \cdot TW_0}{F_0}$, which is the initial asset allocation, consistent with Equation (3).

(iv) The glide path described by Equation (4) is monotonically decreasing in time: as $\frac{e^{r_f(T-t)}-1}{e^{r_f T}-1}$ is positive and decreasing in t for all $t < T$, the denominator of Equation (4) increases with t, implying that w_F is monotonically decreasing in t.

(v) Equation (4) describes the *average* optimal asset allocation, averaged across different realizations of the stock return process. In other words, this is the glide path to be followed if one ignores adjustments to stochastic market fluctuations (i.e. it corrects systematic bias (B2) discussed above, but not bias (B1)).

In order to illustrate the systematic biases induced by linear glide paths, we simulate the wealth dynamics of an investor following the typical glide path shown in Figure 1. We assume that the individual enters the job market at age 20, and keeps receiving labor income (or pension) until age 90. For simplicity, we assume that saving is constant over time, and hence the value of HC is an annuity with a terminal date at age 90. In order to track the stochastic equity component of the portfolio, we simulate stock returns based on the empirical annual return distribution. Stock returns are drawn randomly (with replacement) from the historical annual U.S. market returns over the 1927–2018 period. The risk-free rate is taken as the average T-bill rate over the same period, which is 3.4%. An important parameter in the analysis is the initial ratio between human capital and the total capital, i.e. $\frac{HC_0}{TW_0}$. This ratio determines the initial allocation to stocks in the financial portfolio (see Equation (3)). Figure 2 illustrates the case where initially human capital and financial capital are equal, $\frac{HC_0}{F_0} = 1$, i.e. the case $\frac{HC_0}{TW_0} = 0.5$ (recall that $F + HC = TW$). In the general analysis reported in Tables 1 and 2, different initial ratios are considered. 100,000 different "life-histories" are simulated.

Panel A of Figure 2 shows the actual allocation to equities in the *total* portfolio, w_{TOT}, of an investor following the typical glide path of Figure 1. The thin lines represent five different life-history realizations, i.e. 5 different realizations of 70 stochastic annual stock market returns corresponding to ages 20–90 of the investor. Each of these is calculated as follows: every year a stock annual return is randomly drawn, and the value of the financial portfolio F is updated accordingly. The human capital component decreases deterministically, and w_{TOT} is calculated by Equation (1), where w_F is deterministically determined by the linear glide path shown in Figure 1. Note that w_{TOT} fluctuates because of market fluctuations, as per systematic bias (B1) above: to keep w_{TOT} constant, w_F

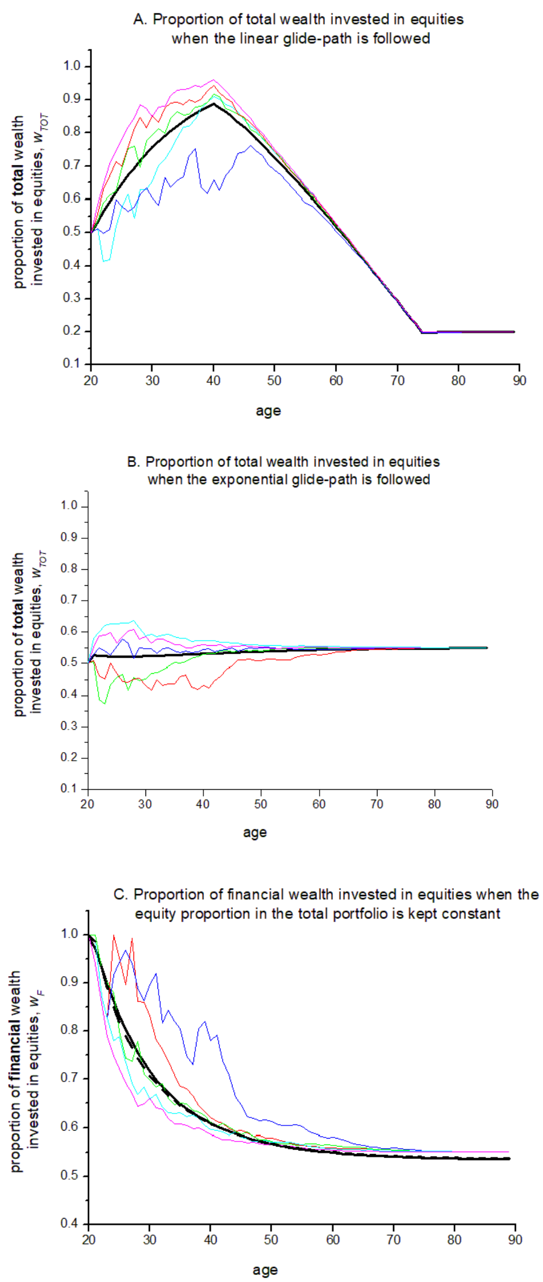

Figure 2 Investment proportion in equities as a function of age.

Figure 2 (*Continued*) Panel A: The proportion in equities in the *total* portfolio (including human capital), w_{TOT}, when the investor follows the linear glide path shown in Figure 1. The thin lines represent five different life-history realizations. The bold line represents the average allocation, averaged over 100,000 life histories. It is evident that the allocation is not constant over time. Panel B: The proportion in equities in the *total* portfolio, w_{TOT}, when the investor follows the exponential glide path given by Equation (4): five life-history realizations, and the average in bold. w_{TOT} is much less variable, but is not constant, because there are no adjustments to market fluctuations. Panel C: The proportion in equities in the *financial* portfolio, w_F, when w_{TOT} is kept exactly constant throughout the life cycle. Note that in order to keep w_{TOT} constant, w_F must change in response to market fluctuations. The bold line is the average w_F, averaged over 100,000 life histories. The dashed line (which is almost indistinguishable from the bold line) shows the theoretical optimal solution for the average allocation, as given by Equation (4).

should be adjusted in response to market fluctuations. But because the predetermined glide path does not make these adjustments, w_{TOT} fluctuates, and is different from one life-history to another. The bold line in Panel A depicts the average w_{TOT}, averaged over all 100,000 life histories. This reveals systematic bias (B2): because the ratio $\frac{HC}{F}$ decreases on average as an exponential, rather than a linear function of time, the linear glide path implies that *on average* the allocation to equities is too high in early years and too low in later years. The average allocation to stocks in the total portfolio, across all 100,000 life histories, is 0.536 (i.e. 53.6%).

When comparing the linear glide path with the exponential glide path we want to compare strategies with the same average allocation, that differ only in the shape of the glide path. Thus, we choose an exponential glide path with the same average allocation to equities in the total portfolio, i.e. with $c = 0.536$. Panel B shows the actual allocation to equities in the *total* portfolio, w_{TOT}, of an investor following the optimal exponential glide path of given by Equation (4) with the relevant parameters: $T = 90$, $c = 0.536$, $\frac{HC_0}{TW_0} = 0.5$, $r_f = 0.033$, and $r_m = 0.095$ (these are the

Table 1 The gain from exact constant allocation relative to the linear glide path (%).

	Initial weight of human capital in total portfolio, $\frac{HC_0}{TW_0}$								
CRRA preferences, γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	6.8	7.1	7.5	7.5	7.8	7.8	7.9	7.8	8.0
1.0	15.1	15.3	15.4	15.6	15.8	15.7	15.6	15.3	15.3
1.5	24.7	24.2	23.7	23.7	23.4	23.1	22.6	22.0	21.8
2.0	35.4	33.3	31.5	30.9	29.9	29.2	28.4	27.5	27.3
3.0	57.8	49.0	42.3	39.8	36.9	35.4	34.0	33.1	33.7
4.0	75.8	59.1	46.8	41.9	37.5	35.0	33.2	32.6	34.3
Negative-exponential preferences, b									
1	13.4	13.0	12.6	12.3	11.6	10.5	9.3	8.4	8.5
10	20.3	19.3	18.2	17.2	16.0	14.8	13.7	13.3	14.8

The certainty equivalent gain achieved by switching from the deterministic linear glide path shown in Figure 1 to exact constant allocation of the portfolio, taking account of the decreasing value of human capital, and adjusting the allocation in the financial portfolio to market fluctuations (i.e. both biases (B1) and (B2) are corrected). The CE gain of Equation (5) is reported in percent, for different preferences and for different values of the weight of human capital in the total portfolio (including human capital and financial wealth).

Table 2 The gain from an exponential glide path relative to the linear glide path (%).

	Initial weight of human capital in total portfolio, $\frac{HC_0}{TW_0}$								
CRRA preferences, γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	6.5	6.0	5.4	4.8	4.3	3.2	2.1	0.4	0.1
1.0	14.3	13.3	12.4	11.2	10.3	8.8	7.2	5.0	1.6
1.5	23.5	21.5	19.8	17.9	16.4	14.3	12.1	9.5	5.8
2.0	33.9	30.3	27.1	24.3	22.0	19.4	16.5	13.6	9.6
3.0	56.4	46.7	38.8	34.0	30.0	26.7	22.7	19.5	15.7
4.0	74.5	58.3	45.4	39.3	33.6	30.0	25.5	22.3	19.2
Negative-exponential preferences, b									
1	13.2	12.5	12.0	11.5	10.9	10.0	9.1	8.4	8.3
10	20.0	18.6	17.2	15.8	14.3	12.7	11.2	10.0	9.7

The certainty equivalent gain, when the allocation is not exactly constant, but instead the allocation follows the exponential glide path given by Equation (4). Note that in this case there are no adjustments to market fluctuations, i.e. bias (B2) is corrected, but bias (B1) is not. Adjustments to market fluctuations improve welfare (every number in Table 1 is larger than the corresponding number in Table 2). The importance of adjustments to market fluctuations depends on the weight of human capital relative to total wealth: it is highest for investors with a large human capital component.

continuously-compounded values of the empirical average returns). The thin lines represent five different life-history realizations (the same five realizations corresponding to those shown in Panel A), and the bold line represents the average over 100,000 histories. The exponential glide path eliminates the systematic "hump" in w_{TOT}: compare the bold lines in Panels A and B. However, there is still some variability in w_{TOT} in each life-history realization, because no adjustments are made in response to market fluctuations. This variability shrinks with time, as the HC component becomes smaller (in the extreme case of $HC = 0$ no adjustments are needed at all, because the total wealth is equal to the financial wealth).

Panel C of Figure 2 shows w_F when w_{TOT} is kept exactly constant. Once again, the 5 thin lines represent five life-history realizations. Each of these is calculated as follows: every year a stock return is randomly drawn, the value of the financial portfolio F is updated accordingly, and the human capital component decreases deterministically. w_F is then calculated by Equation (3), so as to keep the allocation in the portfolio constant at $w_{TOT} = c = 0.536$. The bold line depicts the average w_F, averaged over all 100,000 life histories. The dashed line (which is practically indistinguishable from the bold line) represents the theoretical optimal solution given by Equation (4), with the corresponding parameters detailed above. This excellent fit between the average simulation result and the theoretical Equation (4) implies that the analytical solution derived in continuous-time provides an excellent approximation for the discrete-time setting in the simulations, where returns and allocation adjustments are annual.

4 The Welfare Increase Obtained by Switching from Linear to Exponential

What is the economic benefit that can be achieved by switching from the linear glide path to the exponential glide path, and by adjustments to market fluctuations? For each strategy (linear glide path, exponential glide path, and exponential glide path + adjustment to fluctuations) we calculate the expected utility of terminal wealth, and the certainty equivalent (CE), based on all the 100,000 simulated life histories. We then measure the relative CE gain obtained by the exponential strategies (with and without adjustments to market fluctuations) relative to the linear strategy:

$$\Delta CE \equiv \frac{CE_{exponential}}{CE_{linear}} - 1. \qquad (5)$$

Obviously, the gain in terms of certainty equivalent depends on the assumed preference, namely on the assumed utility function. We calculate the relative CE gain for a wide range of preferences, and different values of the initial $\frac{HC_0}{TW_0}$ ratio. Tables 1 and 2 report the results. Table 1 compares the linear glide path with the optimal strategy of keeping w_{TOT} exactly constant, as in Equation (3), i.e. with adjustments in response to market fluctuations. Table 2 compares the linear glide path with the exponential glide path, with no adjustments to market fluctuations.

Two patterns are evident in Table 1. First, the benefit for correcting the biases increases with the degree of risk aversion. This is to be expected: constant allocation reduces the variation in the possible values of terminal wealth. The second general trend revealed in Table 1 is the increase in benefit as the human capital component decreases (in most cases). This too makes sense: in the extreme case of zero human capital there is no justification whatsoever for the linear glide path employed (Figure 1). When the human capital component is large, decreasing the allocation to equity is helpful, but as Table 1 reveals, the linear glide path does this in a sub-optimal way.

Table 1 shows that for a CRRA investor with a relative risk aversion coefficient of $\gamma = 2$ and a low level of initial human capital, $\frac{HC_0}{TW_0} = 0.1$, the cost of following a linear glide path over the 70-year life cycle is about 35%. This translates to about 0.4% annually.

Table 2 reports the certainty equivalent gain when switching from the linear glide path to the exponential glide path, given by Equation (4) (and shown by the dashed line in Panel B of Figure 2). In this case, there are no adjustments to market fluctuations, and thus w_{TOT} is not exactly constant throughout the life cycle (as shown by the thin lines in Panel B of Figure 2). Indeed, the certainty equivalent in this case is lower than in the case shown in Table 1, where w_{TOT} is kept exactly constant: every number in Table 1 is larger than the corresponding number in Table 2. The extra gain by adjusting to market fluctuations depends on the size of the human capital component: it is most substantial for investors with a high $\frac{HC_0}{TW_0}$ ratio. This is to be expected: the lower the HC component, the less the adjustments are needed (and in the extreme case of $HC = 0$ no adjustments are needed at all). For investors with a large initial component of financial wealth (a low $\frac{HC_0}{TW_0}$ ratio) almost all of the gain is obtained by employing the exponentially decreasing glide path.

Tables 1 and 2 reveal that the welfare gain varies over a very wide range, depending on the individual's preferences and her initial $\frac{HC_0}{TW_0}$ ratio. Of course, for other utility functions the magnitude of the welfare gain may be different, but the principle is the same—by reducing the variations in the asset allocation in the total portfolio risk is reduced, and welfare increases for all risk averters.

What preferences are most realistic? There is considerable debate in the literature regarding this question. Clearly, preferences are heterogeneous, and may be more complex than can be captured by a simple mathematical utility function (see, for example, Diecidue *et al.*, 2015). However, if one is to choose a single utility function that best describes preferences, the empirical and experimental evidence suggests that this should probably be the CRRA preference with γ, the coefficient of relative risk aversion, in the range of 1–2 (see, for example, Arrow, 1971; Dolde and Tobin, 1971; Blume and Friend, 1975; Kydland and Prescott, 1982; Mehra and Prescott, 1985; Levy and Levy, 2020), or even higher (Benninga and Protopapadakis, 1991). Regarding the initial proportion of human capital in the total portfolio, $\frac{HC_0}{TW_0}$, there is no doubt that there is large dispersion of this ratio across the population. Some individuals start their adult lives with almost no financial wealth, i.e. with $\frac{HC_0}{TW_0}$ close to 1. Others inherit (or are expected to inherit) large financial wealth, and have $\frac{HC_0}{TW_0}$ close to 0. To get a very rough idea about the "typical" value of $\frac{HC_0}{TW_0}$, we can look at some empirical statistics, regarding income, savings, and net worth. The Federal Reserve reports the following income distribution for ages 18–29: 67.1% of the individuals in this age group have an annual income below \$40,000, 24.5% have income between \$40,000 and \$100,000, and 8.3% have income above \$100,000.[4] According to the U.S. Bureau of Economic Analysis, the saving rate in the US (which was a double digit figure in the 1980s) was 7.6% in 2018.[5] With the average risk-free rate of 3.4% and the overly simplifying assumption that these parameters are constant over time, the corresponding values for HC_0 are \$80,802 for an income of \$40,000, and \$202,006 for an income of \$100,000.[6] The Federal Reserve reports that the average net worth of people who are 35 years old or less is \$73,500 (unfortunately, the age groups in these two reports do not perfectly coincide). This gives a value of $\frac{HC_0}{TW_0} = 0.52$ for an investor with an income of \$40,000, and $\frac{HC_0}{TW_0} = 0.73$ for an investor with an income of \$100,000.[7] Thus, the most typical range of parameters seems to be CRRA preferences with relative risk aversion γ in the range 1–2, and $\frac{HC_0}{TW_0}$ in the range 0.5–0.8. The shaded cells in Tables 1 and 2 correspond to these parameter ranges. For this range of typical parameter values, the welfare increase by switching from a linear glide path to the exponentially decreasing glide path given by

Table 3 Exponential glide paths for various levels of aggressiveness and initial human capital.

	Defensive, $c = 0.4$			Moderate, $c = 0.5$			Aggressive, $c = 0.6$		
glide pathage	$\frac{HC_0}{TW_0} = 0.3$	$\frac{HC_0}{TW_0} = 0.5$	$\frac{HC_0}{TW_0} = 0.7$	$\frac{HC_0}{TW_0} = 0.3$	$\frac{HC_0}{TW_0} = 0.5$	$\frac{HC_0}{TW_0} = 0.7$	$\frac{HC_0}{TW_0} = 0.3$	$\frac{HC_0}{TW_0} = 0.5$	$\frac{HC_0}{TW_0} = 0.7$
20	57.1	80.0	133.3	71.4	100.0	166.7	85.7	120.0	200.0
25	51.3	63.2	82.3	63.6	77.6	99.7	75.7	91.6	116.1
30	47.7	54.7	64.1	58.9	66.9	77.3	70.0	78.6	89.8
35	45.3	49.7	55.0	56.0	60.8	66.6	66.5	71.6	77.6
40	43.7	46.6	49.9	54.0	57.1	60.6	64.2	67.4	70.9
45	42.6	44.5	46.6	52.7	54.7	56.9	62.8	64.8	67.0
50	41.8	43.1	44.5	51.9	53.2	54.6	61.8	63.1	64.5
55	41.2	42.1	43.0	51.2	52.1	53.0	61.2	62.0	62.9
60	40.8	41.4	42.0	50.8	51.4	52.0	60.8	61.3	61.8
65	40.6	40.9	41.3	50.5	50.9	51.3	60.5	60.8	61.1
70	40.4	40.6	40.9	50.3	50.6	50.8	60.3	60.5	60.7
75	40.2	40.4	40.5	50.2	50.3	50.5	60.2	60.3	60.4
80	40.1	40.2	40.3	50.1	50.2	50.2	60.1	60.1	60.2
85	40.0	40.1	40.1	50.0	50.1	50.1	60.0	60.1	60.1
90	40.0	40.0	40.0	50.0	50.0	50.0	60.0	60.0	60.0

The table shows the allocation to equities in the fund (%) as a function of age for various degrees of aggressiveness, as measured by the allocation to equities in the total portfolio, c. For each level of c, three cases of the initial weight of human capital in the total portfolio are reported. The glide paths are calculated by Equation (4), with $T = 90$, $r_f = 0.033$, and $r_m = 0.095$ (these are the continuously-compounded values of the empirical average risk-free rate and market return, respectively). Note that at the target date (age 90) human capital diminishes to zero, and the allocation to equities in the financial portfolio is equal to the allocation in the total portfolio, c. The higher the initial weight of human capital, the more the allocation to equities in the financial portfolio changes throughout the life cycle. Note that Equation (4) may imply an allocation to equities exceeding 100%, especially in cases of high aggressiveness and a high initial human-capital component. In practical applications the weight to equity can be capped at 100%.

Equation (4) is between 5% and 22%. The welfare increase obtained by keeping the asset allocation exactly constant (i.e. by also adjusting to market fluctuations) is between 15% and 30%.

These results suggest a very easy way to improve target-date funds: simply replace the linear glide path with the exponentially decreasing path given by Equation (4). This simple change, which is almost trivial to implement, is expected to increase the welfare of typical young individuals who enter the job market by 5%–22%, as shown by the shaded cells in Table 2. Annually adjusting the asset allocation to market fluctuations can further substantially increase welfare.

From a practical perspective, the target-date fund should determine the exponentially decreasing glide path by three parameters: (i) the target date T (in the same way it is determined today); (ii) the desired constant allocation to equities in the total portfolio, c, which determines the "aggressiveness" of the fund; and (iii) the typical $\frac{HC_0}{TW_0}$ ratio most relevant for the fund's clientele. Table 3 shows the allocation to equities as a function of age under the exponential glide path (Equation (4)), for various levels of desired "aggressiveness" (as captured by the allocation to stocks, c), and the initial relative weight of human capital.

5 Conclusion

Target-date funds, also called lifecycle funds, offer a valuable service to investors: by reducing the allocation to equities over time, they offset the decline in human capital, and in doing so they help reduce the variation in the asset allocation in the total portfolio throughout the lifecycle. However, the deterministic linear glide paths that they employ lead to two systematic biases relative to the optimal constant asset allocation. The first bias is due to the absence of adjustments to market fluctuations (which change the weight of the financial portfolio relative to the human capital component). The second bias is due to the linear reduction in the allocation to equities.

We suggest a very simple way to correct these biases: replace the linear glide path with the exponential one, and adjust the asset allocation to compensate for market fluctuations. Adjustments need not be more frequent than annual. These easy modifications are expected to increase investors' welfare by 15–30%. By only switching to the exponential glide path, with no adjustments to market fluctuations, welfare typically increases by 5–22%.

It is our hope that exponential glide paths will be widely adopted, and replace linear glide paths as the industry standard. Individuals who manage their portfolios on their own can also enjoy a large increase in welfare by following the exponential glide path.

Notes

1 https://www.morningstar.com/articles/983199/the-best-target-date-series

2 This definition is not standard, as human capital typically refers to the present value of labor income, and is separate from consumption. In our context, the relevant value is income minus consumption, i.e. savings. For example, if consumption is equal to income, i.e. there are no savings, the individual's financial portfolio is his total portfolio.

3 This is the typical assumption in most of the literature. Note, however, that if individuals have the flexibility to increase or decrease their labor, this introduces an option-like component human capital, that may imply an increase in the optimal allocation to stocks with age (see, for example, Bodie *et al*., 1992, 2004, Gomes *et al*., 2008). For simplicity, in the present analysis we do not model the flexibility of labor.

4 See, https//fedralresrve.gov/publication/2017-economic-well-being-of-us-householders, in 2016-income-savings.htm

5 https://www.gov/data/income-saving/personal-saving-rate

[6] Recall our definition of HC as the present value of all future savings.
$\frac{40{,}000 \cdot 0.076}{.034}\left[1-\frac{1}{1.034^{70}}\right]=80,802.$
$\frac{100{,}000 \cdot 0.076}{.034}\left[1-\frac{1}{1.034^{70}}\right]=202,006.$

[7] Of course, the initial income is most likely correlated with the initial net worth, so one would expect the ratio $\frac{HC_0}{TW_0}$ to be actually lower than 0.73 for the individuals with income of $100,000.

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